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Abstract: The design of sliding mode (SM) comprises the selection of a sliding manifold on the state space and a switching logic. The sliding manifold design is associated with the desired dynamics and closed loop specifications, whereas the switching logic is designed to drive and keep the state on the prescribe manifold. The classical design can lead to over or underestimation of the sliding domain, the closed loop robustness and the necessary control power. Here the design of SM is addressed from the global optimization approach using interval arithmetic. A solution to the analysis and synthesis problems of SM design is provided, where the necessary and sufficient conditions are fulfilled in a guaranteed way. For the analysis problem the proposed methodology allows checking sliding mode behaviour over given state domain and parameter sets. For the synthesis problem, the methodology allows designing the sliding manifold and switching logic with a given optimization criterion. The methodology is illustrated with a concluding example.

Keywords: Sliding Modes, Robust control design, Global optimization, Interval analysis.

1. INTRODUCTION

The application of SM to the control of non linear system is well known. Several examples of application could be found in the literature (Khalil (2002), Sira-Ramrez (1993), Rosendo et al. (2016)). The behaviour of this kind of control is composed of two phases. The first phase consists in reaching the sliding surface, and the second one in sliding over it (Utkin et al. (2009)). The design of this control requires choosing an adequate switching function and an appropriate sliding surface, in accordance with the desired dynamics and the SM establishing conditions.

Knowledge of state and parameter excursions are essential in the SM control design. Usually the states and parameters imperfectly known are estimated by their maximal values, and then the resulting control design is tested through simulation. However, even a great number simulations from different initial conditions cannot prove in a guaranteed way that the control law satisfied the sliding condition over the state space. In addition, the design parameters chosen by the operator may not be optimal with respect to criteria such as energy consumption.

In this paper, a control design method based on global optimization and interval analysis techniques is proposed. As a result we derive a method to check in a guaranteed way if the SM necessary and sufficient conditions are fulfilled over bounded state domain and parameter ranges. Concerning the synthesis SM problem, our methodology provides an optimized design based on a given criterion (such as minimal energy consumption, or maximal possible dynamic of the system). Contrary to stochastic methods Wu et al. (2012); Niu et al. (2005); Li et al. (2014) which model uncertain systems as a finite set of deterministic ones, our approach enables to consider continuous uncertainties i.e. to consider an infinite set of systems. However, our method does not applies to time delay systems.

Interval analysis has been applied in the context of SM by Rauh and Aschemann (2012) and Senkel et al. (2014). Their objectives were to obtain online controllers based on interval arithmetic, where the control amplitude is continuously adapted. On the other hand, our approach uses the traditional SM design but we add robustness to the design and give guarantees of the proposed solutions for any initial condition over the given domain.

This paper is organized as follows: Section 2 recalls the SM theory and its associated analysis and synthesis problems. Section 3 introduces global optimization tools and formulates analysis and synthesis problems as optimization ones. Section 4 illustrates our approach with an example. Finally, Section 5 provides some comments and future works.

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2. SM CONTROL THEORY

The sliding modes were originally developed for dynamic systems whose essential open-loop behavior can be modeled with ordinary differential equations (Utkin et al. (2009)). In these systems, it is possible to determinate a robust closed-loop dynamics by applying a discontinuous control action. According to the sign of a switching function, the control signal can take one of two different values, leading to a discontinuous control law with an associated manifold on the state-space (sliding surface). The idea is to enforce the state to reach the prescribed sliding surface and then to slide on it through a very fast switching action. Once this particular mode of operation is established, known as sliding mode, the prescribed manifold imposes the new and desired system dynamics. Among other attractive features sliding regimes are easy to implement, reduce the order of the system dynamics, and provide robustness to matched uncertainties and external disturbances.

The design procedure consists of two stages. First, the equation of the manifold where the system slides is selected in accordance with some performance criterion for the desired dynamics. Then, the discontinuous control should be found such that the system states reach the manifold and sliding mode exists on this manifold.

In order to present the theory, let us consider the dynamic system:

$$\begin{cases} \dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x})u \\ y = h(\mathbf{x}) \end{cases} \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the system state, u is the control signal, y is the output system, and $f(\mathbf{x})$, $g(\mathbf{x})$, $h(\mathbf{x})$ are vector fields in $\mathbb{R}^n$. The variable structure control law is defined as

$$u = \begin{cases} u^- & if \quad \sigma(\mathbf{x}) < \mathbf{0} \\ u^+ & if \quad \sigma(\mathbf{x}) > \mathbf{0} \end{cases} \tag{2}$$

according to the sign of the auxiliary output $\sigma(\mathbf{x})$. The sliding surface S is defined as the manifold where the auxiliary output, also called switching function, vanishes:

$$S = \{\mathbf{x} \in \mathbb{R}^n \mid \sigma(\mathbf{x}) = \mathbf{0}\}. \tag{3}$$

As a result of the switching policy in (2), the reaching condition

$$\begin{cases} \dot{\sigma}(\mathbf{x}) < \mathbf{0} & if \quad \sigma(\mathbf{x}) > \mathbf{0} \\ \dot{\sigma}(\mathbf{x}) > \mathbf{0} & if \quad \sigma(\mathbf{x}) < \mathbf{0} \end{cases} \tag{4}$$

locally holds on both sides of the surface, a switching sequence of very high frequency (ideally infinite) occurs, constraining the system state trajectory to slide on S.

For sliding motion to exist on S (i.e. for satisfying condition (4)), the auxiliary output $\sigma(\mathbf{x})$ must have unitary relative degree with respect to the discontinuous signal, i.e. its first derivative must explicitly depend on u (Utkin et al. (2009)).

Also, it is possible to define the ideal sliding mode using the equivalent control concept. Taking the invariant conditions over the SM surface, we get:

$$\begin{cases} \sigma(\mathbf{x}) = 0 \\ \dot{\sigma}(\mathbf{x}) = \frac{d\sigma}{d\mathbf{x}}\dot{\mathbf{x}} = L_{f+gu_{eq}}\sigma = L_f\sigma + L_g\sigma u_{eq} = 0 \end{cases} \tag{5}$$

where the generic operator $L_f h(x) : \mathbb{R}^n \to \mathbb{R}$ (directional or Lie derivative) denotes the derivative of a scalar field $h(x) : \mathbb{R}^n \to \mathbb{R}$ in the direction of a vector field $f(x) : \mathbb{R}^n \to \mathbb{R}^n$

$$L_f h(x) = \frac{\partial h}{\partial x} f(x). \tag{6}$$

From (5) is possible to obtain $u_{eq}(\mathbf{x})$ a soft control law which makes S an invariant subset.

$$u_{eq}(\mathbf{x}) = -\frac{L_f\sigma}{L_g\sigma} \tag{7}$$

Following this approach is possible to arrive to the necessary and sufficient condition for the SM. It is observed in (7) that $L_g\sigma \neq 0$ is necessary for the existence of u_{eq} and, therefore of SM. Furthermore, a necessary and sufficient condition for the local existence of the sliding mode over S can be derived from (4) and (5). If we consider (without loss of generality) $u^+ > u^-$ it must hold:

$$u^-(\mathbf{x}) < u_{eq}(\mathbf{x}) < u^+(\mathbf{x}) \tag{8}$$

From (8), $u_{eq}(\mathbf{x})$ can be interpreted as an average control action between the maximal and minimal of the system.

From the control designer point of view, it is possible to divide the SM control design into two separate problems:

Problem 1. *SM analysis problem: Given a desired sliding surface $\sigma(\mathbf{x}, \mathbf{k})$ with $\mathbf{x}$ states of the system and $\mathbf{k}$ a vector of fixed tuning parameters (σ with relative degree one with respect to the discontinuous signal u). Verify if u_{eq} fulfills condition given by (8).*

Problem 2. *SM synthesis problem: Given a system with constrained control actions (u^+ and u^-), and an expression of $\sigma(\mathbf{x}, \mathbf{k})$ with $\mathbf{x}$ states of the system and $\mathbf{k}$ a vector of free tuning parameters (σ with relative degree one with respect to the discontinuous signal u). Find the best possible sliding surface σ ($\mathbf{k}$ values) according to a design criterion, which fulfill condition given by (8).*

3. GLOBAL OPTIMIZATION APPROACH

Let us consider a continuous constrained optimization problem formulated as:

$$\begin{cases} \min\limits_{\mathbf{x} \in \mathbb{R}^n} & f(\mathbf{x}) \\ \text{subject to} & C(\mathbf{x}) \leq 0, \end{cases} \tag{9}$$

where f is the objective function which maps $\mathbb{R}^n$ into $\mathbb{R}$, $\mathbf{x} \in \mathbb{R}^n$ is the optimization variable, and C is a function that maps $\mathbb{R}^n$ into $\mathbb{R}$ used to define a subset of $\mathbb{R}^n$ in which the solution is searched. The solution, also called the minimizer, is denoted as $\mathbf{x}^*$ and is the point where f is minimum over the set defined by $\{\mathbf{x} \in \mathbb{R}^n, C(\mathbf{x}) \leq 0\}$. The minimum is denoted as $f^* = f(\mathbf{x}^*)$. From the definition of the minimum, Property (10) holds.

$$\forall \mathbf{x} \in \mathbb{R}^n \text{ such as } C(\mathbf{x}) \leq 0,\ f(\mathbf{x}) \geq f^*. \tag{10}$$

If f and C are not convex functions, local optimization techniques have no warranty to converge to the global solution $\mathbf{x}^*$. On the other hand, global optimization methods converge to the global minimum and provide an enclosure $[\underline{f^*}, \overline{f^*}]$ of f^*. One well-known technique from global optimization is the Branch and Bound algorithm based on interval arithmetic Kearfott (1992).

3.1 Branch and Bound based on interval arithmetic

In order to present the Interval Branch and Bound Algorithm (IBBA), several definitions must be given.

Definition 1. *An interval $\boldsymbol{x}$ is a closed connected subset of $\mathbb{R}$ (Moore et al. (2009)), described by its endpoints $\underline{x}$ and $\overline{x}$:*

$$\boldsymbol{x} = [\underline{x}, \overline{x}] = \{x \mid \underline{x} \leq x \leq \overline{x}\},$$
$$\text{with } \underline{x} \in \mathbb{R} \cup \{-\infty\} \text{ and } \overline{x} \in \mathbb{R} \cup \{+\infty\}$$

The set of real intervals is denoted by $\mathbb{IR}$ and the set of n-dimensional interval vectors, also called boxes, is denoted by $\mathbb{IR}^n$.

Definition 2. *Let $\boldsymbol{x} \in \mathbb{IR}^n$ be a box. An inclusion function $[f]$ of f maps $\mathbb{IR}^n$ into $\mathbb{IR}$ and respects the following property:*

$$f(\boldsymbol{x}_i) = \{f(x),\, x \in \boldsymbol{x}_i\} \subseteq [f](\boldsymbol{x}_i).$$

Interval arithmetic extends common operators ($+$, $-$, $\times$, *sin*, *cos*, exp, log,...) to $\mathbb{IR}$ and provide inclusion function of most of analytic functions. Let us suppose that inclusion functions of f and C can be defined, and the $\mathbf{x}^*$ is searched in $\mathbb{X} \in \mathbb{IR}^n$. The IBBA computes a guaranteed lower bound $\underline{f}$ and an upper bound $\overline{f}$ of f^*. To do so, IBBA repeatedly bisects $\mathbb{X}$ in smaller boxes $\boldsymbol{x_i}$ and discards them if it is proven that $\mathbf{x}^* \notin \boldsymbol{x_i}$. This happens if the constraint is not satisfied over $\boldsymbol{x}_i$:

$$\begin{aligned} \underline{[C](\boldsymbol{x}_i)} > 0 &\iff \forall \mathbf{x} \in \boldsymbol{x}_i, C(\mathbf{x}) > 0, \\ &\iff \mathbf{x}^* \notin \boldsymbol{x}_i, \end{aligned} \tag{11}$$

or if a feasible point $\tilde{x}$ has been found such that any points in $\boldsymbol{x}_i$ can provide a better feasible solution:

$$\underline{[f](\boldsymbol{x}_i)} > f(\tilde{x}) \geq f^* \iff \mathbf{x}^* \notin \boldsymbol{x}_i. \tag{12}$$

The IBBA stops when the distance between $\underline{f}$ and $\overline{f}$ reaches the desired precision d, with

$$\underline{f} = \min_i \underline{[f](\boldsymbol{x}_i)},\ \overline{f} = f(\tilde{x}) \tag{13}$$

Fig. 1 illustrates IBBA. The box $\boldsymbol{x}_{11}$ is proved not to contain $\mathbf{x}^*$ due to Property (11), as well as boxes $\boldsymbol{x}_{12}$ and $\boldsymbol{x}_{21}$ due to Property (12).
The Set Inversion Via Interval Analysis (SIVIA) algorithm is a branch and bound and allows to approximate the feasible region of $\mathbb{X}$, described by the constraints, by a sub-paving, which is a union of non-overlapping boxes. SIVIA algorithm bisects $\mathbb{X}$ in smaller boxes $\boldsymbol{x}_i$ until the constraint

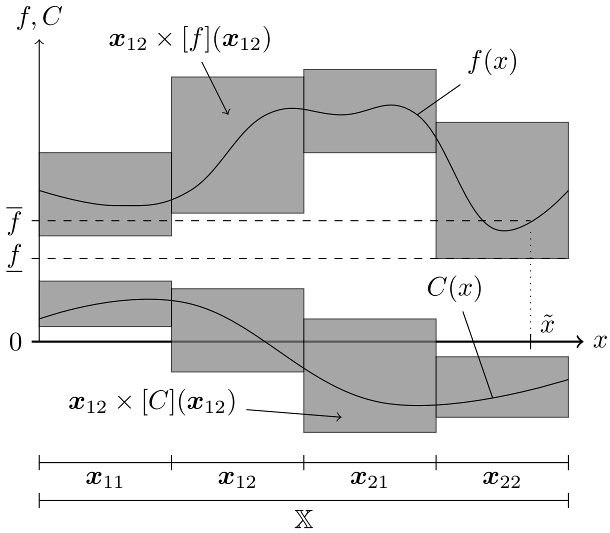

Fig. 1. Illustration of IBBA and SIVIA algorithms.

is proved to be fulfilled over $\boldsymbol{x}_i$ thanks to (14) or not to be fulfilled thanks to (11).

$$\overline{[C](\boldsymbol{x}_i)} \leq 0 \iff \forall \mathbf{x} \in \boldsymbol{x}_i, C(\mathbf{x}) \leq 0 \tag{14}$$

SIVIA algorithm stops when boxes $\boldsymbol{x}_i$ reach a minimum size ϵ. In Figure 1, SIVIA returns the sub-paving made of $\boldsymbol{x}_{11}$, $\boldsymbol{x}_{12}$, $\boldsymbol{x}_{21}$ and $\boldsymbol{x}_{22}$ indicating that $\boldsymbol{x}_{11}$ is not a subset of the feasible set, $\boldsymbol{x}_{22}$ is a subset of the feasible set, and that nothing could be proved for $\boldsymbol{x}_{12}$ and $\boldsymbol{x}_{21}$. That is, $\boldsymbol{x}_{11}$ is an inner approximation the feasible set and $\boldsymbol{x}_{11} \cup \boldsymbol{x}_{12} \cup \boldsymbol{x}_{21}$ is an outer approximation. These approximations can be improved by bisecting $\boldsymbol{x}_{12}$ and $\boldsymbol{x}_{21}$ in smaller boxes.
Finally, IBBA has $[f]$, $[C]$, $\mathbb{X}$ and d as inputs and provides a feasible point $\tilde{\mathbf{x}}$ and a guaranteed enclosure $[\underline{f}, \overline{f}]$ of the global minimum f^*. SIVIA algorithm has $[C]$, $\mathbb{X}$ and ϵ as inputs and provides a sub-paving which characterizes the feasible region.

3.2 Analysis problems

Let us consider the analysis problem defined in Section 2. This problem can be rewritten as (15), where IBBA can provide an enclosure $[\underline{u_{eq}(\theta)}, \overline{u_{eq}(\theta)}]$ of the minimum $u_{eq}(\theta)^*$. Being θ the vector of tuning parameters given by the operator, δ a set of variable parameters and Δ a subset of $\mathbb{IR}^{n\delta}$ with n_δ the dimension of δ.

$$\left\{ \min_{\delta \in \Delta} \max(u_{eq}(\theta, \delta) - u^-, -u_{eq}(\theta, \delta) + u^+) \right. \tag{15}$$

We will show how IBBA can be used to ensure that θ is a feasible solution of this constraint satisfaction problem (CSP).

From Property (10), we can derive Property (16) and Property (17)

$$\begin{aligned} &\underline{u_{eq}(\theta)} > 0 \\ &\iff u_{eq}(\theta)^* > 0 \\ &\iff \forall \delta \in \Delta, \max(u_{eq}(\theta, \delta) - u^-, -u_{eq}(\theta, \delta) + u^+) > 0 \\ &\iff \forall \delta \in \Delta, u_{eq}(\theta, \delta) - u^- > 0 \text{ and } , -u_{eq}(\theta, \delta) + u^+ > 0 \\ &\iff \forall \delta \in \Delta, u^- < u_{eq}(\theta, \delta) \text{ and } , u_{eq}(\theta, \delta) < u^+ \end{aligned} \tag{16}$$

$$
\begin{array}{l}
\overline{u_{eq}(\theta)} < 0 \\
\iff u_{eq}(\theta)^* < 0 \Rightarrow u_{eq}(\theta,\delta^*) < 0 \\
\iff \max(u_{eq}(\theta,\delta^*) - u^-, -u_{eq}(\theta,\delta^*) + u^+) < 0 \\
\iff u_{eq}(\theta,\delta^*) - u^- < 0 \text{ or } , -u_{eq}(\theta,\delta^*) + u^+ < 0 \\
\iff \exists\delta \in \Delta, u^- > u_{eq}(\theta,\delta) \text{ or } , u_{eq}(\theta,\delta) > u^+
\end{array} \tag{17}
$$

According to Proposition (16), if $\underline{u_{eq}(\theta)} > 0$, θ is a feasible solution to Problem (1), which ensures that the system will slide on the sliding surface S over the subset Δ. According to Proposition (17), if $\overline{u_{eq}(\theta)} < 0$, θ is a not feasible solution to Problem 1, which means that the system will not slide over S in all Δ. Actually, the system will leave S at least at δ^* the solution to Problem (15).

If $0 \in [\underline{u_{eq}(\theta)}, \overline{u_{eq}(\theta)}]$, it is not possible to prove whether or not θ is a feasible solution. In the case where θ is not a feasible solution, SIVIA algorithm can be used to characterize the largest subset of Δ where the sliding condition is established. That is, the set:

$$
\left\{\delta \in \Delta | u_{eq}(\theta,\delta) < u^+ \text{ and } u^- < u_{eq}(\theta,\delta)\right\}
$$

can be approximated by a sub-paving.

3.3 Synthesis problems

Synthesis problems consist either in characterizing the set of feasible tuning parameters with respect to SM conditions and let the operator choose θ in this set, or in minimizing a given cost function over this feasible set. SIVIA algorithm and IBBA are suited to perform such computation.

Let Θ be a subset of $\mathbb{IR}^{n_\theta}$, $f : \mathbb{R}^{n_\theta} \mapsto \mathbb{R}$ be a cost function given by the system designer, and C^*_θ be the minimum of Problem (15) with θ fixed. We suppose that an inclusion function of f is available. Let $C^* : \mathbb{R}^{n_\theta} \mapsto \mathbb{R}$ be the function that maps θ into C^*_θ. The synthesis problem can be expressed in a general way as the optimization problem (18).

$$
\begin{cases}
\min\limits_{\theta\in\Theta} \; f(\theta) \\
\text{s.t.} \quad C^*(\theta) \le 0
\end{cases} \tag{18}
$$

The constraint of Problem (18) implies the resolution of an analysis problem over Δ. Using interval analysis, it is possible to provide an enclosure of C^* over a box $\boldsymbol{\theta}$ Monnet et al. (2016). As a consequence IBBA can be used to solve Problem (18) and SIVIA to characterized the set defined by the constraint:

$$
\{\theta \in \Theta, C^*(\theta) < 0\} \tag{19}
$$

More generally, such a constraint is called a Semi Infinite Constraint (SIC), since it is equivalent to the infinite set of constraint $C(\theta,\delta) \le 0$, $\forall\delta \in \Delta$, but involves only a finite number of variables. Optimization problems involving SIC are called Semi Infinite Programs (SIP) and can be solved in a global way with different methods Mitsos (2011); Bhattacharjee et al. (2005), and the characterization of the set defined by SICs has been studied in several works Goldsztejn et al. (2009); Ratschan (2002).

4. CASE STUDY

In this section we illustrate the application of the proposed approach to a non-linear system. This is a simplified version of the angular control of a satellite based on the Cayley-Rodriguez parameter.

The system behaviour is modeled in a simplified way by the following equations:

$$
\begin{cases}
\dot{x}_1 = \frac{1}{2}(1+x_1^2)x_2 \\
\dot{x}_2 = \frac{1}{J}u
\end{cases} \tag{20}
$$

where the parameters involved are:

- x_1 Cayley-Rodrigues parameter to define orientation.
- x_2 angular velocity.
- u control action.
- J system inertia.

Assuming it is desired to impose a closed loop dynamics given by:

$$
\dot{x}_1 = \lambda(x_1 - r) \tag{21}
$$

with r the position reference and λ an approaching rate tuning parameter. Then, we can propose a sliding mode control with: $u = sign(\sigma)$ and a sliding surface of the form:

$$
\Sigma = \left\{ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad : \quad \sigma(\mathbf{x}) = -x_2 - \frac{2\lambda(x_1 - r)}{1+x_1^2} = 0 \right\} \tag{22}
$$

It is possible to observe that the necessary condition for SM is fulfilled:

$$
L_g\sigma = \frac{\partial\sigma}{\partial x_1}\frac{\partial\sigma}{\partial x_2}\begin{bmatrix} 0 \\ \frac{1}{J} \end{bmatrix} = \frac{1}{J} \neq 0 \tag{23}
$$

And given σ it is possible to find u_{eq} as:

$$
u_{eq} = \frac{2J\lambda^2(x_1 - r)(-x_1^2 + 2rx_1 + 1)}{(1+x_1^2)^2} \tag{24}
$$

Resulting the necessary and sufficient condition for the SM:

$$
u^- < u_{eq} < u^+ \tag{25}
$$

In the following, we pose this system according to a given operating condition in the form of the problems explained in Section 2, and an analysis of the results is made.

Example 1. *Analysis SM:*

Given $u^+ = 1$, $u^- = -1$, $J = 1$, $\lambda = 0.5$, $r = 1$ we desire to know if the selected configuration results in a satisfactory sliding behaviour. This means to solve the problem establish by (15). In this case, we could establish the following relations to the problem as:

$$
\begin{cases}
\theta \;\leftrightarrow\; \textit{no variable: fixed by the operator} \\
\delta \;\leftrightarrow\; x_1 \\
\Delta \;\leftrightarrow\; [-5, 5]
\end{cases} \tag{26}
$$

As a result we get the enclosure of the global minimum

$$\max(u_{eq}(\theta,\delta^*) - u^-, -u_{eq}(\theta,\delta^*) + u^+) \in [-0.526874, -0.516874]$$

proving satisfactory that it is a good choice for the domain Δ *tested.*

Remark : *From this result, it is possible to conclude that* $u^+ = -u^- = 0.526874$ *is the smallest value of the control input such that the sliding condition holds over* $x_1 \in [-5, 5]$. *Choosing this value of control input over the initial value* $u^+ = -u^- = 1$ *will result in energy consumption savings. In addition, with IBBA one can compute the global minimum of* u_{eq} *over* Δ *and also the global maximum. These two values correspond to the greatest value of* u^- *and the lowest value of* u^+ *which ensure the sliding condition, respectively.*

Example 2. *Synthesis SM:*

Given $u^+ = 1$, $u^- = -1$, $J = 1$, *we desire to know what is the highest possible value of* λ *that result in a satisfactory sliding behaviour for a position reference in* $[-1, 1]$. *This can be done by solving Problem (18). In this case, we can establish the variable relation to the problem as:*

$$\begin{cases} \theta \leftrightarrow \lambda \\ \Theta \leftrightarrow [0, 5] \\ \delta \leftrightarrow (x_1, r)^T \\ \Delta \leftrightarrow ([-10, 10], [-1, 1])^T \end{cases} \tag{27}$$

The objective function is given by $f : \lambda \to -\lambda$ *in order to have a minimization problem. The IBBA algorithm provides* $[-0.70, -0.69]$ *as an enclosure of the minimum. The best feasible point found, with respect to the sliding condition, is* $\lambda = 0.69$. *In addition it is guaranteed that no value of* λ *greater than* 0.7 *exists such as the sliding condition holds over* Δ.

Example 3. *Synthesis SM:*

Given $u^+ = 1$, $u^- = -1$, $J = 1$, *we now want to know which are the possible values of* λ *and* r *that result in a satisfactory sliding behaviour. This means to solve the problem established by (19). In this case, we can establish the variable relation to the problem as:*

$$\begin{cases} \theta \leftrightarrow (\lambda, r)^T \\ \Theta \leftrightarrow ([0, 5], [-5, 5])^T \\ \delta \leftrightarrow x_1 \\ \Delta \leftrightarrow [-5, 5] \end{cases} \tag{28}$$

As a result we get the sub-paving of Fig. 2, where red boxes imply no satisfaction of the conditions imposed, green boxes satisfaction of them, and finally blue boxes indicate that the algorithm cannot determine the conditions. One can remark that the solution of the synthesis problem proposed for Example 2 is consistent with the sub-paving of Fig. 2, since the subset defined by $\{(\lambda, r), \lambda = 0.69$ *and* $r \in [-1, 1]\}$ *belongs to the union of the green and blue sets.*

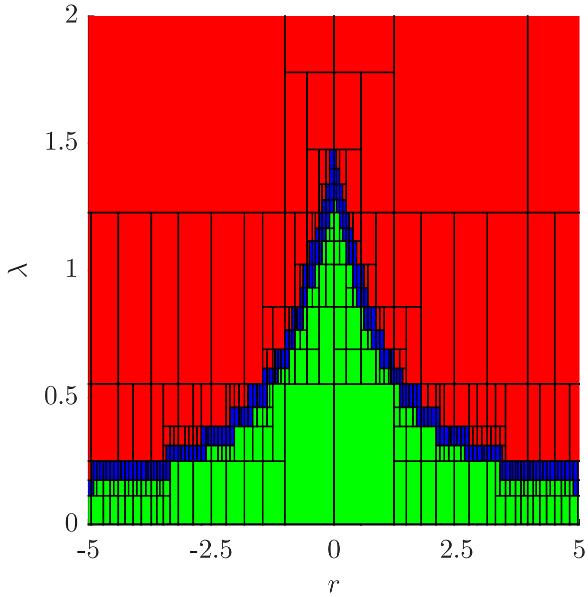

Fig. 2. SM guaranteed existence domain for Example 3

In Fig. 3, it is possible to see the system behaviour with different sliding surfaces (different λ *values) for a reference* $r = 2$. *Notes from Fig. 2 that for* $r = 2$ *over* $\lambda = 0.4$ *the SM condition is no longer satisfied with the given* Δ, *which is also verified by the blue trajectory of Fig.3 which leaves the sliding surface once it is reached.*

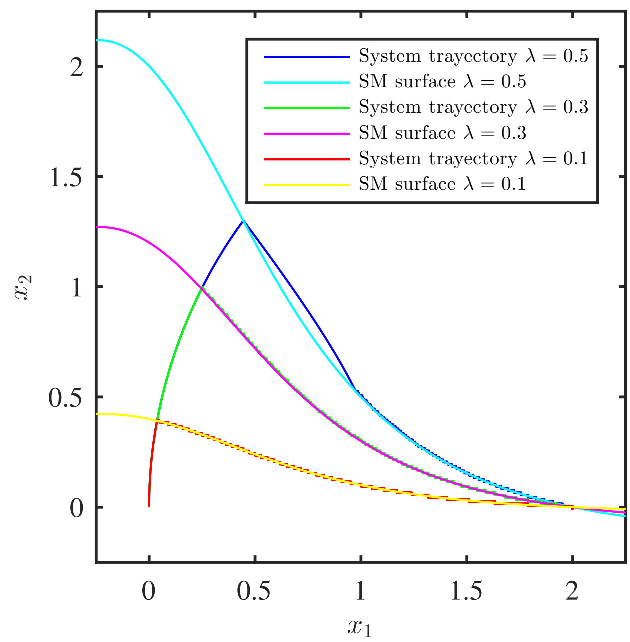

Fig. 3. SM surfaces for different λ values

Remark : *Although the reference signal cannot be really considered as a tuning variable, it is the case in this example. Doing so, we get Figure 2 which indicates for which values of* r *the SM condition holds given a value of* λ. *Variables that would normally belong to* δ *can be changed as tuning variables to provide additional information about the system.*

Example 4. *Synthesis SM:*

Given the same conditions of Example 3, now, we are concerned about the parameter uncertainty over the system, and its effect over the SM conditions. Here it is considered a J parameter variation bounded in the interval $[0.5, 1.5]$*. In this case we can also establish the problem as in (19). In this case, we can establish the variable relation to this problem as:*

$$\begin{cases} \theta & \leftrightarrow (\lambda, r)^T \\ \Theta & \leftrightarrow ([0,5], [-5,5])^T \\ \delta & \leftrightarrow (x_1, J)^T \\ \Delta & \leftrightarrow ([-5,5], [0.5, 1.5])^T \end{cases} \tag{29}$$

As a result, we get the sub-paving shown in Fig. 4 where it is possible to see how the area satisfying the SM conditions is smaller than in Fig. 2 due to the uncertainty of J.

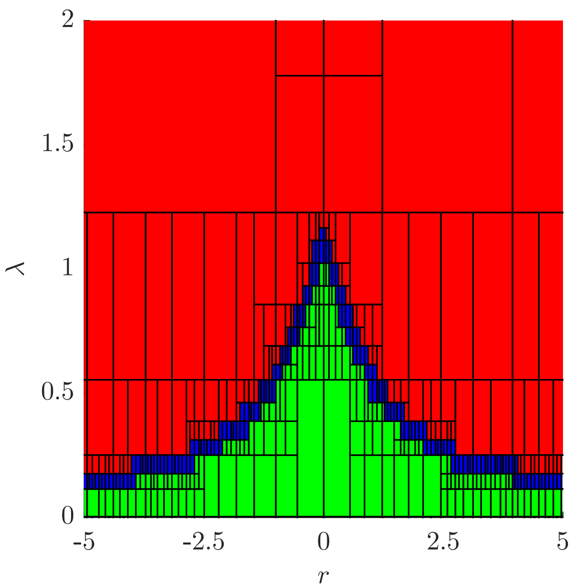

Fig. 4. SM guaranteed existence domain for Example 4

5. CONCLUSIONS

The chosen approach presented in this work shows to be an efficient method to complement the traditional SM control design. Using the interval analysis tools to solve a non-convex global optimization problem, our approach optimizes the SM design for a given criterion. Furthermore, it adds robustness and guarantees the SM set up in front of the process variations and the constrained analyzed state space. To do so, global optimization methods must be used since the synthesis and analysis problems are not convex contrary to the problems emerging in the stochastic approaches which are generally formulated as linear matrix inequalities (therefore convex). The complexity of IBBA grows exponentially with the number of variables, and may fails to solve very large problems.

A particular point to mention is the construction of the sub-paving graphics as design tools. They allow not only to know a particular solution but also to know which is the domain, with respect to the analyzed variables, where the solution is valid.

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