# Chain of set inversion problems; Application to reachability analysis

Benoît Desrochers, Luc Jaulin

# Chain of set inversion problems; Application to reachability analysis

Benoît Desrochers[1,2] and Luc Jaulin[2]

***Abstract*—This paper deals with the set inversion problem $\mathbb{X} = \mathbf{f}^{-1}(\mathbb{Y})$ in the case where $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ depends on a parameter vector $\mathbf{p} \in \mathbb{R}^q$ which is known to be inside a box $[\mathbf{p}]$. We show that for a large class of problems, we can obtain an accurate approximation of the solution set, without bisecting in the p-space. To do this, symbolic methods are required to cast our initial problem into a chain of set-inversion problems, the links of which have some nice properties with respect to p. As an application, we consider the problem of computing the set of all initial states of an uncertain discrete-time state system that reach a target set $\mathbb{Y}$ in a given time.**

## I. Introduction

Reachability analysis is a classical problem in control theory [1] [2][3][4] and has several applications, for instance *(i)* to validate some properties of cyber-physic systems [5][6], *(ii)* to ensure the safe configuration during the landing [7] or *(iii)* to avoid collisions [8] with other aircrafts. Reachability analysis allows us to guarantee that the system with a given control law will always reach a target [9]. In this paper, we deal with the problem of computing the set $\mathbb{X}_0$ of all initial states $\mathbf{x}_0 = \mathbf{x}(0)$, of a discrete time system such that at time $\bar{k}$ the state $\mathbf{x}(\bar{k})$ is inside a given set $\mathbb{Y}$.

We assume that the system is described by

$$\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k)), \tag{1}$$

where $\mathbf{x}(k) \in \mathbb{R}^n$ is the state vector and $\mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^n$ is the evolution function of the system. If we define the function

$$\boldsymbol{\varphi}^k = \underbrace{\mathbf{f} \circ \mathbf{f} \circ \cdots \circ \mathbf{f}}_{k \text{ times}}, \tag{2}$$

we have

$$\mathbb{X}_0 = (\boldsymbol{\varphi}^{\bar{k}})^{-1}(\mathbb{Y}) \tag{3}$$

which corresponds to a set inversion problem [10]. Note that the set inversion problem is considered as much easier than the direct image problem [11]. Equivalently, we could have written

$$\mathbb{X}_0 = \mathbf{f}^{-1}(\mathbb{X}_1), \mathbb{X}_1 = \mathbf{f}^{-1}(\mathbb{X}_2), \dots, \mathbb{X}_{\bar{k}} = \mathbb{Y} \tag{4}$$

and the computation of $\mathbb{X}_0$ amounts to solve a chain of set inversion problems. In both cases, the problem can be solved using a *Set Inversion* approach [10] based on interval analysis. Set inversion and interval analysis are more and more used in the context of nonlinear estimation [12] [13] [14] [15] [16].

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Assume now that there exists an unknown input vector $\mathbf{u}(k) \in [\mathbf{u}]$ in the system which may correspond to a control or a perturbation. We now have

$$\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k), \mathbf{u}(k)). \tag{5}$$

Thus the initial feasible set $\mathbb{X}_0$ becomes also uncertain: it depends on the sequence $\mathbf{u}(k)$. Define the function $\boldsymbol{\varphi}^k$ as follows.

$$\begin{array}{rcl} \boldsymbol{\varphi}^1(\mathbf{x}(0), \mathbf{u}(0)) & = & \mathbf{f}(\mathbf{x}(0), \mathbf{u}(0)) \\ \boldsymbol{\varphi}^{k+1}(\mathbf{x}(0), \mathbf{u}(0:k)) & = & \mathbf{f}(\boldsymbol{\varphi}^k(\mathbf{x}(0), \mathbf{u}(0:k-1)), \mathbf{u}(k)) \end{array}$$

where $\mathbf{u}(0:k-1)$ denotes the sequence $\{\mathbf{u}(0), \dots, \mathbf{u}(k-1)\}$. If we define

$$\begin{array}{rcl} \mathbb{X}_0^{\subset} & = & \{\mathbf{x}_0 | \forall \mathbf{u}(0:\bar{k}-1) \in [\mathbf{u}]^{\bar{k}}, \boldsymbol{\varphi}^{\bar{k}}(\mathbf{x}(0), \mathbf{u}(0:\bar{k}-1)) \in \mathbb{Y}\} \\ \mathbb{X}_0^{\supset} & = & \{\mathbf{x}_0 | \exists \mathbf{u}(0:\bar{k}-1) \in [\mathbf{u}]^{\bar{k}}, \boldsymbol{\varphi}^{\bar{k}}(\mathbf{x}(0), \mathbf{u}(0:\bar{k}-1)) \in \mathbb{Y}\}, \end{array}$$

we have

$$\mathbb{X}_0^{\subset} \subset \mathbb{X}_0 \subset \mathbb{X}_0^{\supset}. \tag{6}$$

The sets $\mathbb{X}_0^{\subset}$ and $\mathbb{X}_0^{\supset}$ are classically called the *minimal* and the *maximal backward reach set* [17]. Both sets $\mathbb{X}_0^{\subset}$ and $\mathbb{X}_0^{\supset}$ are difficult to compute, mainly due to the fact that existing methods cannot characterize from inside the penumbra [18] which corresponds to the set $\mathbb{X}_0^{\partial} = \mathbb{X}_0^{\supset} \backslash \mathbb{X}_0^{\subset}$. Moreover, even to characterize inside $\mathbb{X}_0^{\subset}$ or outside $\mathbb{X}_0^{\supset}$, existing interval methods will need to bisect inside the box $[\mathbf{u}]^{\bar{k}}$ to control the accuracy of the characterization. This is not satisfactory due to its large dimension. Note that there exists numerical methods [19] to compute an outer approximation of the maximal backward reach set $\mathbb{X}_0^{\supset}$, or an inner approximation of the minimal backward reach set $\mathbb{X}_0^{\subset}$, but the main difficulty in inside the penumbra, *i.e.*, to compute accurately, an inner approximation of $\mathbb{X}_0^{\supset}$ and an outer approximation $\mathbb{X}_0^{\subset}$.

The main contribution of this paper is to show that by rewriting the problem as a chain of set inversion problem, we will be able to characterize, with an arbitrary accuracy the feasible set $\mathbb{X}_0$, without any bisection of $[\mathbf{u}]^{\bar{k}}$. An inner approximation of the penumbra $\mathbb{X}_0^{\partial} = \mathbb{X}_0^{\supset} \backslash \mathbb{X}_0^{\subset}$, which corresponds to the uncertainty of the approximation, will also be computed. This was not possible before except for some specific cases such as when the system is linear [20].

The paper is organized as follows. Section II formulates the set inversion problem in the case where it depends on a parameter vector $\mathbf{p}$ which is assumed to be inside a box $[\mathbf{p}]$. Section III introduces the specific case where the function to be inverted is a link function. This will allow us to build, in Section IV, a large class of set inversion problems that can be solved efficiently, without any bisection inside the parameter space. Test-cases related to reachability analysis are presented in Section V. Section VI concludes the paper.

## II. Problem

Consider the set inversion problem

$$\mathbb{X} = \mathbf{f}^{-1}(\mathbb{Y}) \tag{7}$$

where $\mathbf{f}:\mathbb{R}^n \to \mathbb{R}^m$, and $\mathbb{Y}$ is a subset of $\mathbb{R}^m$. Solving a set inversion problem consists of characterizing the solution set $\mathbb{X}$ from inside and from outside. We consider the uncertain case where $\mathbf{f}$ depends on a parameter vector $\mathbf{p} \in \mathbb{R}^q$. The function should thus be written as $\mathbf{f}(\mathbf{x},\mathbf{p})$, but for simplicity of notation when we will compose functions to form a chain, we will often write $\mathbf{f}_{\mathbf{p}}(\mathbf{x})$. In such a case, the solution set $\mathbb{X}$ also depends on the value of $\mathbf{p}$ and is denoted by $\mathbb{X}(\mathbf{p})$. We have:

$$\mathbb{X}^{\subset} \subset \mathbb{X}(\mathbf{p}) \subset \mathbb{X}^{\supset}, \tag{8}$$

where

$$\begin{array}{lll}\mathbb{X}^{\subset} & = & \bigcap_{\mathbf{p}\in[\mathbf{p}]} \mathbf{f}_{\mathbf{p}}^{-1}(\mathbb{Y}) = \{\mathbf{x} | \forall \mathbf{p} \in [\mathbf{p}], \mathbf{f}_{\mathbf{p}}(\mathbf{x}) \in \mathbb{Y}\} \\ \mathbb{X}^{\supset} & = & \bigcup_{\mathbf{p}\in[\mathbf{p}]} \mathbf{f}_{\mathbf{p}}^{-1}(\mathbb{Y}) = \{\mathbf{x} | \exists \mathbf{p} \in [\mathbf{p}], \mathbf{f}_{\mathbf{p}}(\mathbf{x}) \in \mathbb{Y}\}.\end{array}$$

The set $\mathbb{X}^{\partial} = \mathbb{X}^{\supset} \backslash \mathbb{X}^{\subset}$ is called the *penumbra* [21] and contains the unknown boundary of $\mathbb{X}$.

Denote by $(\mathcal{P}(\mathbb{R}^n), \subset)$ the powerset of $\mathbb{R}^n$ equipped with the inclusion $\subset$ as an order relation. The powerset $\mathcal{P}(\mathbb{R}^n)$ is a complete lattice with respect to $\subset$. The meet operator corresponds to the intersection and the join to the union. The pair $[\mathbb{X}^{\subset}, \mathbb{X}^{\supset}]$ is an interval in $\mathcal{P}(\mathbb{R}^n)$ which is called a *thick set* and denoted by $[\![\mathbb{X}]\!]$. A thick set partitions $\mathbb{R}^n$ into three zones: the clear zone $\mathbb{X}^{\subset}$, the penumbra $\mathbb{X}^{\partial} = \mathbb{X}^{\supset} \backslash \mathbb{X}^{\subset}$ and the dark zone $\mathbb{R}^n \backslash \mathbb{X}^{\supset}$. A thick set $[\![\mathbb{X}]\!]$ is a sub-lattice of $(\mathcal{P}(\mathbb{R}^n), \subset)$, *i.e.*, if $\mathbb{A} \in [\![\mathbb{X}]\!]$,$\mathbb{B} \in [\![\mathbb{X}]\!]$, then $\mathbb{A} \cap \mathbb{B} \in [\![\mathbb{X}]\!]$ and $\mathbb{A} \cup \mathbb{B} \in [\![\mathbb{X}]\!]$.

**Notation**. The set inversion problem we want to solve will be written as

$$[\![\mathbb{X}]\!] = \mathbf{f}_{[\mathbf{p}]}^{-1}(\mathbb{Y}) \tag{9}$$

and will be called a *thick set inversion problem.*

To characterize $[\![\mathbb{X}]\!]$ with an arbitrary accuracy, *i.e.*, to characterize the sets $\mathbb{X}^{\subset}$ and $\mathbb{X}^{\supset}$, existing interval methods, will need to bisect with respect to the $\mathbf{p}$-space, which makes them inefficient. Moreover, they are not able to prove that a box is inside the penumbra. It has been show recently [18] how it can proved that a box is inside the penumbra. It will be shown in Section III that, in the specific case where the function $\mathbf{f}_{\mathbf{p}}(\mathbf{x})$ is a *link* function (*i.e.*, for all $\mathbf{x}$, and for any box $[\mathbf{p}]$, the set $\mathbf{f}_{[\mathbf{p}]}(\mathbf{x})$ is a box), bisections on $[\mathbf{p}]$ can be avoided. We now illustrate how this can be done on a simple example.

**Example 1.** Consider the thick set inversion problem $[\![\mathbb{X}]\!] = \mathbf{f}_{[\mathbf{p}]}^{-1}([\mathbf{y}])$ where

$$\mathbf{f}_{\mathbf{p}}(\mathbf{x}) = \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{10}$$

Assume that $[\mathbf{p}] = [2,3] \times [2,5] \times [4,5] \times [-6,-1]$ and $[\mathbf{y}] = [5,19] \times [-7,11]$. Take a box $[\mathbf{x}] = [0,1] \times [2,3]$ and let us show how is can be proved that it is inside the penumbra. Note that for simplicity, we have taken all intervals with a constant sign, but the method can be made much more general if we use modal interval analysis [22] [23] or symbolic interval arithmetic [24]. Inside $[\mathbf{x}]$, we have

$$\begin{array}{lll}\mathbf{f}(\mathbf{x},[\mathbf{p}]) & = & \begin{pmatrix} [p_1^- x_1 + p_2^- x_2, p_1^+ x_1 + p_2^+ x_2] \\ [p_3^- x_1 + p_4^- x_2, p_3^+ x_1 + p_4^+ x_2] \end{pmatrix} \\ & = & \left[ \begin{pmatrix} p_1^- x_1 + p_2^- x_2 \\ p_3^- x_1 + p_4^- x_2 \end{pmatrix}, \begin{pmatrix} p_1^+ x_1 + p_2^+ x_2 \\ p_3^+ x_1 + p_4^+ x_2 \end{pmatrix} \right] \\ & = & \left[ \mathbf{f}_{[\mathbf{p}]}^-(\mathbf{x}), \mathbf{f}_{[\mathbf{p}]}^+(\mathbf{x}) \right]\end{array}$$

which corresponds to a box. If $\mathbf{x} \in [\mathbf{x}] = [0,1] \times [2,3]$, $\mathbf{f}^-(\mathbf{x},[\mathbf{p}])$ belongs to the box

$$\begin{array}{lll}\left[\mathbf{f}_{[\mathbf{p}]}^-\right]([\mathbf{x}]) & = & \begin{pmatrix} 2\cdot[0,1] + 2\cdot[2,3] \\ 4\cdot[0,1] - 6\cdot[2,3] \end{pmatrix} \\ & = & \begin{pmatrix} [0,2] + [4,6] \\ [0,4] + [-18,-12] \end{pmatrix} \\ & = & \begin{pmatrix} [4,8] \\ [-18,-8] \end{pmatrix}\end{array} \tag{11}$$

and $\mathbf{f}^+(\mathbf{x},[\mathbf{p}])$ belongs to the box

$$\begin{array}{lll}\left[\mathbf{f}_{[\mathbf{p}]}^+\right]([\mathbf{x}]) & = & \begin{pmatrix} 3\cdot[0,1] + 5\cdot[2,3] \\ 5\cdot[0,1] - 1\cdot[2,3] \end{pmatrix} \\ & = & \begin{pmatrix} [0,3] + [10,15] \\ [0,5] + [-3,-2] \end{pmatrix} \\ & = & \begin{pmatrix} [10,18] \\ [-3,3] \end{pmatrix}.\end{array} \tag{12}$$

Since $\left[\mathbf{f}_{[\mathbf{p}]}^-\right]([\mathbf{x}]) \cap [\mathbf{y}] = \emptyset$ and $\left[\mathbf{f}_{[\mathbf{p}]}^+\right]([\mathbf{x}]) \subset [\mathbf{y}]$, we conclude that $[\mathbf{x}]$ is inside the penumbra. This is illustrated by Figure 1.

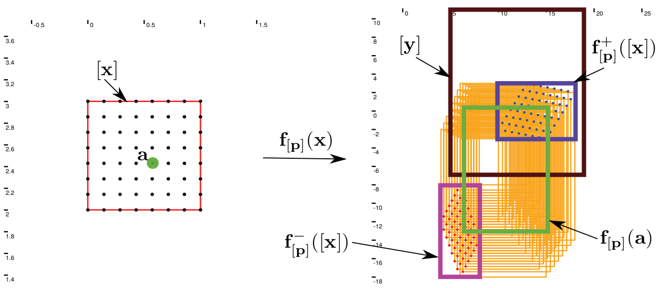


Fig. 1. For $[\mathbf{x}] = [0,1] \times [2,3]$ the set $\mathbf{f}_{[\mathbf{p}]}^+$ (blue dots) is inside $[\mathbf{y}]$ whereas $\mathbf{f}_{[\mathbf{p}]}^-$ (red dots) is outside. We conclude that $[\mathbf{x}]$ is inside the penumbra

In the specific case where the function $\mathbf{f}_{\mathbf{p}}(\mathbf{x})$ is a *link* function (see Section III for more details), for all $\mathbf{x}$, the set $\mathbf{f}_{[\mathbf{p}]}(\mathbf{x})$ is a box, denoted by $\left[\mathbf{f}_{[\mathbf{p}]}^-(\mathbf{x}), \mathbf{f}_{[\mathbf{p}]}^+(\mathbf{x})\right]$, with lower bound $\mathbf{f}_{[\mathbf{p}]}^-(\mathbf{x}) \in \mathbb{R}^m$ and upper bound $\mathbf{f}_{[\mathbf{p}]}^+(\mathbf{x}) \in \mathbb{R}^m$. These two bounds can be described by an algorithm and two inclusion functions can be obtained for them using the rules of interval computation [25]. This means that if $\mathbf{x} \in [\mathbf{x}]$, we can obtain, under the form of an algorithm, two boxes $\left[\mathbf{f}_{[\mathbf{p}]}^-\right]([\mathbf{x}])$ and $\left[\mathbf{f}_{[\mathbf{p}]}^+\right]([\mathbf{x}])$ such that

$$\mathbf{f}_{[\mathbf{p}]}^-([\mathbf{x}]) \in \left[\mathbf{f}_{[\mathbf{p}]}^-\right]([\mathbf{x}]) \text{ and } \mathbf{f}_{[\mathbf{p}]}^+([\mathbf{x}]) \in \left[\mathbf{f}_{[\mathbf{p}]}^+\right]([\mathbf{x}]).$$

To characterize the thick set $[\![\mathbb{X}]\!]$ of the thick set inversion problem $[\![\mathbb{X}]\!] = \mathbf{f}_{[\mathbf{p}]}^{-1}(\mathbb{Y})$, we start with a huge box (large enough to contain $\mathbb{X}^{\supset}$) and we cut it into several other boxes all stored inside a list $\mathcal{L}$. For each box $[\mathbf{x}]$ of $\mathcal{L}$, and in we compute the two boxes $\left[\mathbf{f}_{[\mathbf{p}]}^{-}\right]([\mathbf{x}])$ and $\left[\mathbf{f}_{[\mathbf{p}]}^{+}\right]([\mathbf{x}])$. From these two boxes, we are able to test if $[\mathbf{x}] \subset \mathbb{X}^{\subset}$ or if $[\mathbf{x}] \cap \mathbb{X}^{\supset} = \emptyset$ or if $[\mathbf{x}]$ is included inside the penumbra. If nothing can be concluded, the box $[\mathbf{x}]$ is bisected and the two resulting boxes are stored inside the list. The treatment is applied to all boxes of the list until all boxes become too small (*i.e.*, with a width smaller than a threshold $\varepsilon$). Note that, if we are able to conclude that a box $[\mathbf{x}]$ is inside the penumbra, no bisection would occur.

For Example 1, the method generates Figure 2 (right) which is an approximation of a thick set with the inner part (red), the outer part (blue) and the penumbra (orange). A classical interval method would not able to characterize the penumbra and yields Figure 2 (left). Since the resulting algorithm bisects in the $\mathbf{x}$-sp ace, its complexity is exponential with respect to the dimension of $\mathbf{x}$. Its application is thus limited to low dimensions. If do not bisect on the $\mathbf{p}$-space (as we suggest in this paper using chains and link functions), the complexity with respect to $\mathbf{p}$ can be considered as polynomial.

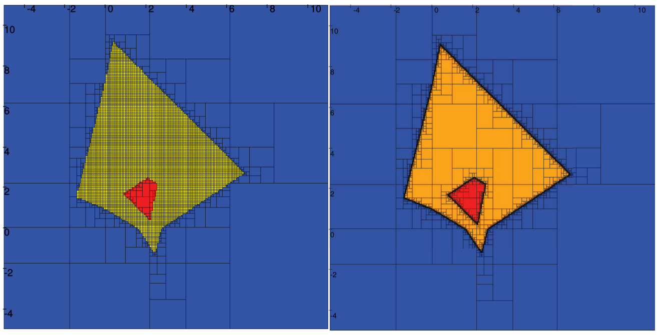

Fig. 2. Left: Classical interval methods accumulate on the yellow thick boundary (the penumbra). Right: the method we propose here allows a fast treatment of the penumbra. The frame box is $[-5, 11]^2$

## III. Link functions

This section considers the case of thick set-inversion problems in the case where the functions to be inverted have some good properties. These functions will be the links to build more complex functions to be inverted.

**Definition 2. Link function.** If, for a box $[\mathbf{p}] \subset \mathbb{R}^q$, the set

$$\mathbf{f}_{[\mathbf{p}]}(\mathbf{x}) = \mathbf{f}(\mathbf{x}, [\mathbf{p}]) = \{\mathbf{y} \in \mathbb{R}^p \mid \exists \mathbf{p} \in [\mathbf{p}],\ \mathbf{y} = \mathbf{f}_{\mathbf{p}}(\mathbf{x})\}$$

is a box, then the function $\mathbf{f}$ is said to be a *link*. Links will be composed later in Section IV in order to build a *chain*. If $m = 1$ then the function $f(\mathbf{x}, \mathbf{p})$ becomes *scalar*. Note that all $f(\mathbf{x}, \mathbf{p})$ that are scalar and continuous with respect to $\mathbf{p}$, are links.

Due to their specific box-shaped structure, link functions can be inverted with respect to $\mathbf{x}$ without bisections with respect to the uncertain parameter box $[\mathbf{p}]$.

**Example 3.** The function

$$f(\mathbf{x}, p) = 20e^{-x_1 p} - 8e^{-x_2 p} \tag{13}$$

is a scalar link function, due to the continuity of $f$ with respect to $p$. Now, to get a computable expression for $f(\mathbf{x}, [p])$, we need to compute the extremum of the function inside the interval $[p] = [p^-, p^+]$. Since

$$\begin{array}{rrcl} & \frac{\partial f(\mathbf{x}, \bar{p})}{\partial p} & = & 0 \\ \Leftrightarrow & -20x_1 \cdot e^{-x_1 \bar{p}} + 8x_2 \cdot e^{-x_2 \bar{p}} & = & 0 \\ \Leftrightarrow & e^{(x_2 - x_1)\bar{p}} & = & \frac{2x_2}{5x_1} \\ \Leftrightarrow & \bar{p} & = & \frac{1}{x_2 - x_1} \ln\left(\frac{2x_2}{5x_1}\right), \end{array} \tag{14}$$

inside the interval $[p] = [p^-, p^+]$, $f(\mathbf{x}, p)$, may have the extremum $\bar{y} = f(\mathbf{x}, \bar{p})$. As a result, if we define $y^- = f(\mathbf{x}, p^-)$, $y^+ = f(\mathbf{x}, p^+)$, we have

$$\mathbf{f}(\mathbf{x}, [p]) = \begin{cases} [\min\{y^-, y^+, \bar{y}\}, \max\{y^-, y^+, \bar{y}\}] & \text{if } \bar{p} \in [p] \\ [\min\{y^-, y^+\}, \max\{y^-, y^+\}] & \text{if } \bar{p} \notin [p] \end{cases}$$

Equivalently,

$$\mathbf{f}(\mathbf{x}, [p]) = \left[\mathbf{f}_{[p]}^{-}(\mathbf{x}), \mathbf{f}_{[p]}^{+}(\mathbf{x})\right] \tag{15}$$

where

$$\mathbf{f}_{[p]}^{-}(\mathbf{x}) = \begin{cases} \min\{y^-, y^+\} & \text{if } \bar{p} \notin [p] \\ \min\{y^-, y^+, \bar{y}\} & \text{if } \bar{p} \in [p] \end{cases} \tag{16}$$

and

$$\mathbf{f}_{[p]}^{+}(\mathbf{x}) = \begin{cases} \max\{y^-, y^+\} & \text{if } \bar{p} \notin [p] \\ \max\{y^-, y^+, \bar{y}\} & \text{if } \bar{p} \in [p] \end{cases} \tag{17}$$

Note that both $\mathbf{f}_{[p]}^{-}(\mathbf{x})$ and $\mathbf{f}_{[p]}^{+}(\mathbf{x})$ belong to $\mathbb{R}$. If $\mathbf{x} \in [\mathbf{x}]$, we have

$$\mathbf{f}_{[p]}^{-}(\mathbf{x}) \in \left[\mathbf{f}_{[p]}^{-}\right]([\mathbf{x}]) = \begin{cases} \min\{[y^-], [y^+]\} & \text{if } [\bar{p}] \cap [p] = \emptyset \\ \min\{[y^-], [y^+], [\bar{y}]\} & \text{otherwise} \end{cases}$$

and

$$\mathbf{f}_{[p]}^{+}(\mathbf{x}) \in \left[\mathbf{f}_{[p]}^{+}\right]([\mathbf{x}]) = \begin{cases} \max\{[y^-], [y^+]\} & \text{if } [\bar{p}] \cap [p] = \emptyset \\ \max\{[y^-], [y^+], [\bar{y}]\} & \text{otherwise} \end{cases}$$

where

$$\begin{array}{rcl} [\bar{p}] & = & \frac{1}{[x_2] - [x_1]} \ln\left(\frac{2[x_2]}{5[x_1]}\right) \\ [\bar{y}] & = & f([\mathbf{x}], [\bar{p}] \cap [p]) \\ [y^-] & = & f([\mathbf{x}], p^-) \\ [y^+] & = & f([\mathbf{x}], p^+). \end{array} \tag{18}$$

**Definition 4. Vector components.** The family of vectors $\{\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_\ell\}$ correspond to *vector components* of the vector $\mathbf{p}$ if each $\mathbf{p}_i$ is a subvector of $\mathbf{p}$ and the indexes are all disjoint. For instance if $\mathbf{p} = (p_1, \dots, p_9)$, then $\{\mathbf{p}_1, \mathbf{p}_2\}$, where $\mathbf{p}_1 = (p_2, p_9)$ and $\mathbf{p}_2 = (p_8, p_1, p_3)$, correspond to vector components of $\mathbf{p}$. Note that the two index sets $\{2, 9\}$ and $\{8, 1, 3\}$ are disjoint.

The fact that index sets are disjoint implies in independencies between parameters. The conservatism related to the *dependency effect*, well known in interval analysis [26], is thus avoided. In practice, this will allow us to avoid bisection in the $\{\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_\ell\}$-space.

**Proposition 5.** *Consider $\ell$ scalar link functions, $f_i(\mathbf{x}, \mathbf{p}_i)$ , where vectors $\{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_\ell\}$ are vector components of the vector $\mathbf{p}$. The function*

$$\mathbf{f}(\mathbf{x}, \mathbf{p}) = \begin{pmatrix} f_1(\mathbf{x}, \mathbf{p}_1) \\ \vdots \\ f_\ell(\mathbf{x}, \mathbf{p}_\ell) \end{pmatrix} \tag{19}$$

*is a link.*

*Proof:* Consider the box $[\mathbf{p}]$ and its box components $[\mathbf{p}_1], \ldots, [\mathbf{p}_\ell]$. The set $\mathbf{f}(\mathbf{x}, [\mathbf{p}])$ is the Cartesian product of $\ell$ intervals:

$$\mathbf{f}(\mathbf{x}, [\mathbf{p}]) = f_1(\mathbf{x}, [\mathbf{p}_1]) \times \cdots \times f_\ell(\mathbf{x}, [\mathbf{p}_\ell]) \tag{20}$$

which corresponds to a box. ∎

**Example 6.** The function

$$\mathbf{f}(\mathbf{x}, \mathbf{p}) = \begin{pmatrix} p_1 x_1 x_2 + p_1 p_2 sin(x_2) \\ p_3 x_1 + p_3 p_4 cos(x_1) x_2 \end{pmatrix} \tag{21}$$

is a link. Now, if we replace $p_3$ by $p_2$, a dependency occurs between the two components of $\mathbf{f}$ and the function is not a link anymore.

**Consequence**. If $\mathbf{f}_\mathbf{p}(\mathbf{x})$ is a link, for all $\mathbf{x}$, the set $\mathbf{f}_{[\mathbf{p}]}(\mathbf{x})$ is a box. If $\mathbf{x} \in [\mathbf{x}]$, the lower and upper bounds of the box $\mathbf{f}_{[\mathbf{p}]}(\mathbf{x})$ is included inside the boxes $\left[\mathbf{f}^-_{[\mathbf{p}]}\right]([\mathbf{x}])$ and $\left[\mathbf{f}^+_{[\mathbf{p}]}\right]([\mathbf{x}])$. As a consequence, the methodology explained in Section II can be applied to find an inner and outer approximations of the thick set $[\![\mathbb{X}]\!] = \mathbf{f}^{-1}_{[\mathbf{p}]}(\mathbb{Y})$, with an approximation of the penumbra.

**Example 7.** Consider the problem of estimating the parameters $q_1$ and $q_2$ of the model

$$y(\mathbf{q}, t) = 20e^{-q_1 t} - 8e^{-q_2 t}.$$

We assume that 10 measurements $y_i$ have been collected at time $t_i$. The uncertainties on the pair $(t_i, y_i)$ is represented by intervals as represented by the table below and Figure 3 (left). We are interested by the thick set

$$[\![\mathbb{Q}]\!] = \left\{\mathbf{q} \in \mathbb{R}^2 | \forall i, y(\mathbf{q}, t_i) \in [y_i]\right\}.$$

If we set

$$\begin{aligned} \mathbf{f}_{[\mathbf{t}]}(\mathbf{q}) &= \begin{pmatrix} y(\mathbf{q}, [t_1]) \\ \vdots \\ y(\mathbf{q}, [t_{10}]) \end{pmatrix} \\ [\mathbf{y}] &= [y_{10}] \times \cdots \times [y_{10}] \\ [\mathbf{t}] &= [t_1] \times \cdots \times [t_{10}] \end{aligned}$$

then, we have

$$[\![\mathbb{Q}]\!] = \mathbf{f}^{-1}_{[\mathbf{t}]}([\mathbf{y}]),$$

which is thick set inversion problem with the link function $\mathbf{f}_{[\mathbf{t}]}(\mathbf{q})$ to be inverted. The inversion yields the approximation of Figure 3 (right).

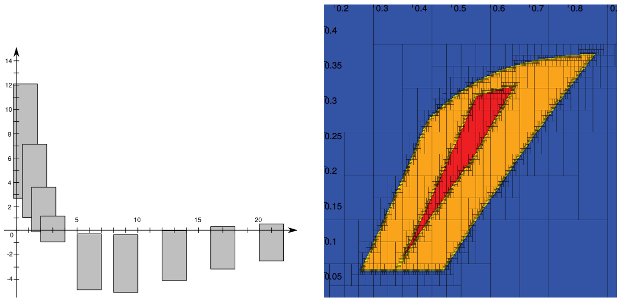

Fig. 3. Left: measurement boxes $[t_i] \times [y_i]$. Right: Approximation of the thick set $[\![\mathbb{Q}]\!]$ in the parameter space

| $i$ | $[t_i]$ | $[y_i]$ |
|---|---|---|
| 1 | [0.25, 1.25] | [2.7,12.1] |
| 2 | [1, 2] | [1.04,7.14] |
| 3 | [1.75, 2.75] | [-0.13,3.61] |
| 4 | [2.5, 3.5] | [-0.95,1.15] |
| 5 | [5.5, 6.5] | [-4.85,-0.29] |
| 6 | [8.5, 9.5] | [-5.06,-0.36] |
| 7 | [12.5, 13.5] | [-4.1,-0.04] |
| 8 | [16.5, 17.5] | [-3.16,0.3] |
| 9 | [20.5, 21.5] | [-2.5,0.51] |
| 10 | [24.5, 25.5] | [-2,0.6] |

## IV. Chain

**Definition 8. Chain**. The function $\varphi(\mathbf{x}, \mathbf{p}) = \varphi_\mathbf{p}(\mathbf{x})$ is said to be a *chain* if it can be written as

$$\varphi_\mathbf{p}(\mathbf{x}) = \mathbf{f}^\ell_{\mathbf{p}_\ell} \circ \cdots \circ \mathbf{f}^2_{\mathbf{p}_2} \circ \mathbf{f}^1_{\mathbf{p}_1}(\mathbf{x}) \tag{22}$$

where $\{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_\ell\}$ are vector components of $\mathbf{p}$ and the $\mathbf{f}^k_{\mathbf{p}_k}$ are links.

Note that the parameter vectors $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_\ell$ are independent. This implies that the pessimism due to the dependency effect will not take place during the resolution. Moreover, the fact that $\mathbf{f}^k_{\mathbf{p}_k}$ are links implies that pessimism due to the wrapping effect (of a set by a box) will not exist.

**Example 9.** The function

$$\varphi_\mathbf{p}(\mathbf{x}) = \begin{pmatrix} sin(p_3 x_1^2) + p_4 \\ \frac{p_5(p_1 e^{x_2} + p_1 p_2 x_1 x_2)}{p_3 x_1^2} \end{pmatrix} \tag{23}$$

is not a link. Indeed, $p_3$ occurs on both components of $\varphi_\mathbf{p}$. Now, if we define

$$\mathbf{f}^1_{\mathbf{p}_1}(\mathbf{x}) = \begin{pmatrix} p_1 e^{x_2} + p_1 p_2 x_1 x_2 \\ p_3 x_1^2 \end{pmatrix} \tag{24}$$

$$\mathbf{f}^2_{\mathbf{p}_2}(\mathbf{z}) = \begin{pmatrix} sin(z_2) + p_4 \\ \frac{p_5 z_1}{z_2} \end{pmatrix} \tag{25}$$

with $\mathbf{p}_1 = (p_1, p_2, p_3)$ and $\mathbf{p}_2 = (p_4, p_5)$ then, we have

$$\varphi_\mathbf{p}(\mathbf{x}) = \mathbf{f}^2_{\mathbf{p}_2} \circ \mathbf{f}^1_{\mathbf{p}_1}(\mathbf{x}). \tag{26}$$

Since both $\mathbf{f}^2_{\mathbf{p}_2}, \mathbf{f}^1_{\mathbf{p}_1}$ are links, $\varphi_\mathbf{p}(\mathbf{x})$ is a chain.

**Consequence**. Solving a thick set inversion problem $[\![\mathbb{X}]\!] = \varphi^{-1}_{[\mathbf{p}]}(\mathbb{Y})$ in the case where $\varphi$ is a chain can be done by solving several thick set inversion problems, without any bisection inside the box $[\mathbf{p}]$. Rewriting a chain as a composition of link functions can probably be done by symbolic methods, but we do not know any approach or algorithm to perform this task. Of course, we should not allow the dimension of the intermediate spaces to be arbitrary large compare to the dimension of $\mathbf{x}$ if we want to be efficient.

The following section provides some test-cases of the thick inversion of chains, applied to reachability problems.

## V. Test-cases

**Test-case 1**. Consider the linear system

$$\mathbf{x}(k+1) = \mathbf{A} \cdot \mathbf{x}(k) \tag{27}$$

where

$$\mathbf{A}(k) \in \begin{pmatrix} [2.5, 3] & [2, 3] \\ [4, 4.5] & [-3, -2] \end{pmatrix} \tag{28}$$

plays the role of the parameter vector $\mathbf{p}$. Assume that we want to reach the target

$$\mathbb{Y} = [4, 20] \times [-8, 12] \tag{29}$$

at time $\bar{k} = 3$. Solving the chain of thick set inversion problems yields the approximation of the thick sets $[\![\mathbb{X}]\!](0), [\![\mathbb{X}]\!](1), [\![\mathbb{X}]\!](2), [\![\mathbb{X}]\!](3)$ as depicted on Figure 4. Note that $[\![\mathbb{X}]\!](3)$ corresponds to the box $\mathbb{Y}$. We first compute the thick set $[\![\mathbb{X}]\!](2) = \mathbf{f}^{-1}_{[\mathbf{A}]}([\![\mathbb{X}]\!](3))$. Note that $[\![\mathbb{X}]\!](3)$ is here a thin set (*i.e.*, a classical set) whereas $[\![\mathbb{X}]\!](2)$ is thick (*i.e.*, with a penumbra). Then we compute $[\![\mathbb{X}]\!](1) = \mathbf{f}^{-1}_{[\mathbf{A}]}([\![\mathbb{X}]\!](2))$ and finally, we compute $[\![\mathbb{X}]\!](0) = \mathbf{f}^{-1}_{[\mathbf{A}]}([\![\mathbb{X}]\!](1))$. At each step, the penumbra inflates, due to the accumulation of uncertainties.

The frame boxes are $[-0.6, 0.8] \times [-0.4, 1]$ for $[\![\mathbb{X}]\!](0)$, $[-1.5, 3] \times [-2.5, 2.2]$ for $[\![\mathbb{X}]\!](1)$, $[-4, 8] \times [-3, 10]$ for $[\![\mathbb{X}]\!](2)$, and $[-5, 27] \times [-15, 20]$ for $[\![\mathbb{X}]\!](3)$.

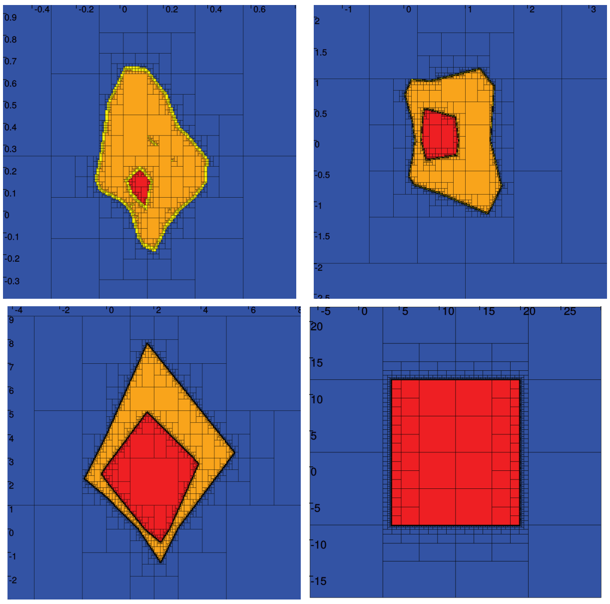

Fig. 4. Approximation of the sets $\mathbb{X}(0), \mathbb{X}(1), \mathbb{X}(2), \mathbb{X}(3)$

**Test-case 2**. Consider the nonlinear system

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{pmatrix} x_1(k) + x_2^2(k) \cdot u_1(k) \\ \frac{1}{2} \cdot x_1(k) \cdot x_2(k) + u_2(k) \end{pmatrix} \tag{30}$$

with $u_1 \in [1, 2]$ and $u_2 \in [-2, -1]$. The set $\mathbb{Y}$ is assumed to be a centred disk which has to be reached at time $\bar{k} = 3$. Solving the chain of thick set inversion problem generates Figure 5. The frame boxes are $[-12, 10] \times [-12, 10]$ for all $[\![\mathbb{X}]\!](i)$.

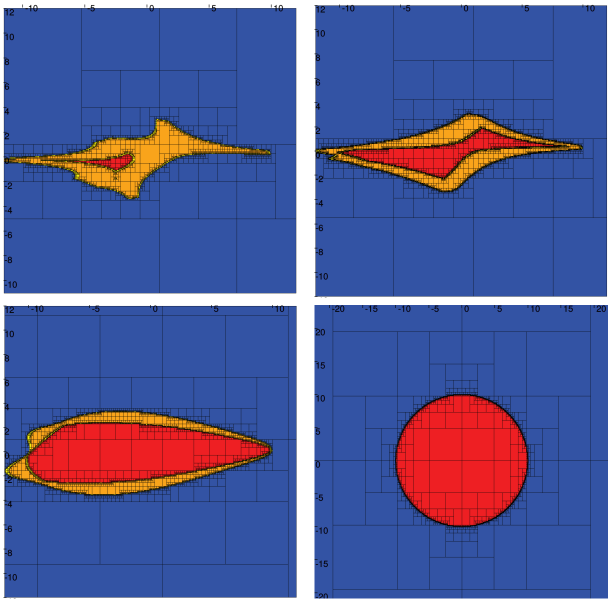

Fig. 5. Approximation of the sets $\mathbb{X}(0), \mathbb{X}(1), \mathbb{X}(2), \mathbb{X}(3)$

## VI. Conclusion

In this paper, we have presented a new interval based approach to compute the set $\mathbb{X}_0$ of initial states that will reach a target $\mathbb{Y}$ in a finite time. The dynamic system that is considered is discrete time and uncertain (the uncertainties are represented by intervals). The principle is to transform the problem into a chain of set inversion problems composed with links, *i.e.*, a function with nice properties for set inversion.

Among the problems that remain to be solved, we need (1) to find symbolic methods for the decomposition of a chain into links with low dimensions; (2) to extend the approach to continuous systems described by differential equations and to study how the penumbra can be characterized from inside in such a case; and (3) to solve the chain of set inversion problems in parallel and not sequentially as done here.

## References


[1] J. Aubin and H. Frankowska, *Set-Valued Analysis*. Birkhäuser, Boston, 1990.

[2] F. Blanchini and S. Miani, *Set-Theoretic Methods in Control*. Springer Science & Business Media, Oct. 2007.

[3] N. Ramdani and N. Nedialkov, "Computing Reachable Sets for Uncertain Nonlinear Hybrid Systems using Interval Constraint Propagation Techniques," *Nonlinear Analysis: Hybrid Systems*, vol. 5, no. 2, pp. 149–162, 2011.

[4] O. Bouissou, A. Chapoutot, A. Djaballah, and M. Kieffer, "Computation of parametric barrier functions for dynamical systems using interval analysis," in *IEEE CDC*, Los Angeles, United States, 2014.

[5] M. Konecny, W. Taha, J. Duracz, A. Duracz, and A. Ames, "Enclosing the behavior of a hybrid system up to and beyond a zeno point," in *Cyber-Physical Systems, Networks, and Applications (CPSNA)*, 2013.

[6] W. Taha and A. Duracz, "Acumen: An open-source testbed for cyber-physical systems research," in *CYCLONE'15*, 2015.

[7] A. M. Bayen, I. M. Mitchell, M. K. Osihi, and C. J. Tomlin, "Aircraft Autolander Safety Analysis Through Optimal Control-Based Reach Set Computation," *Journal of Guidance, Control, and Dynamics*, vol. 30, no. 1, pp. 68–77, 2007.

[8] A. Desilles, H. Zidani, and E. Cruck, "Collision analysis for an UAV," in *AIAA Guidance, Navigation, and Control Conference*. American Institute of Aeronautics and Astronautics, 2012, pp. 13–16.

[9] J.-P. Aubin, "Viability Kernels and Capture Basins of Sets Under Differential Inclusions," *SIAM Journal on Control and Optimization*, vol. 40, no. 3, pp. 853–881, Jan. 2001.

[10] L. Jaulin and E. Walter, "Guaranteed nonlinear parameter estimation via interval computations," *Interval Computation*, pp. 61–75, 1993.

[11] O. Mullier, E. Goubault, M. Kieffer, and S. Putot, "General inner approximation of vector-valued functions," *Reliable Computing*, vol. 18, no. 0, pp. 117–143, 2013.

[12] C. Jauberthie, N. Verdière, and L. Travé-Massuyès, "Set-membership identifiability: definitions and analysis," in *In Proceedings of the 18th IFAC World Congress, Milan, Italie*, vol. 00. aaa, 2011, pp. 12 024–12 029.

[13] T. Raissi, N. Ramdani, and Y. Candau, "Set membership state and parameter estimation for systems described by nonlinear differential equations," *Automatica*, vol. 40, pp. 1771–1777, 2004.

[14] T. Raissi, D. Efimov, and A. Zolghadri, "Interval state estimation for a class of nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 57, no. 1, pp. 260–265, 2012.

[15] V. Drevelle and P. Bonnifait, "igps: Global positioning in urban canyons with road surface maps," *IEEE Intelligent Transportation Systems Magazine*, vol. 4, no. 3, pp. 6–18, 2012.

[16] J. A. dit Sandretto, G. Trombettoni, D. Daney, and G. Chabert, "Certified calibration of a cable-driven robot using interval contractor programming," in *Computational Kinematics, Mechanisms and Machine Science*, ser. Spinger, F. Thomas and A. P. Gracia, Eds., 2014.

[17] L. Jaulin and E. Walter, *Modeling Techniques for Uncertain Systems*. Birkhäuser, 1994, ch. Set Inversion, with Application to Guaranteed Nonlinear Estimation and Robust Control, pp. 3–20.

[18] B. Desrochers and L. Jaulin, "Computing a guaranteed approximation the zone explored by a robot," *IEEE Transaction on Automatic Control*, vol. 62, no. 1, pp. 425–430, 2017.

[19] I. M. Mitchell, "Application of level set methods to control and reachability problems in continuous and hybrid systems," 2003.

[20] A. Girard, "Computation and stability analysis of limit cycles in piecewise linear hybrid systems," in *1st IFAC Conference on Analysis and Design of Hybrid Systems*, 2003, pp. 181–186.

[21] W. E. L. Grimson and T. Lozano-Pérez, "Recognition and localization of overlapping parts in two and three dimensions," in *IEEE International Conference on Robotics and Automation*, 1985, pp. 61–66.

[22] A. Goldsztejn, D. Daney, M. Rueher, and P. Taillibert, "Modal intervals revisited: a mean-value extension to generalized intervals," in *Proceedings of the Quantification in Constraint Programming, Barcelona, Spain, 2005 (QCP 2005)*, 2005.

[23] P. H. Vinas, M. A. Sainz, J. Vehi, and L. Jaulin, "Quantified set inversion algorithm with applications to control," *Reliable computing*, vol. 11, no. 5, pp. 369–382, 2006.

[24] L. Jaulin and G. Chabert, "Resolution of nonlinear interval problems using symbolic interval arithmetic," *Engineering Applications of Artificial Intelligence*, vol. 23, no. 6, pp. 1035–1049, 2010.

[25] R. B. Kearfott, *Rigorous Global Search: Continuous Problems*. Dordrecht, the Netherlands: Kluwer, 1996.

[26] R. E. Moore, *Methods and Applications of Interval Analysis*. Philadelphia, PA: SIAM, 1979.