Thick set inversion

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Abstract

This paper deals with the set inversion problem $\mathbb{X} = \mathbf{f}^{-1}(\mathbb{Y})$ in the case where the function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ and the set $\mathbb{Y}$ are both uncertain. The uncertainty is treated under the form of intervals. More precisely, for all $\mathbf{x}$, $\mathbf{f}(\mathbf{x})$ is inside the box $[\mathbf{f}](\mathbf{x})$ and the uncertain set $\mathbb{Y}$ is bracketed between an inner set $\mathbb{Y}^{\subset}$ and an outer set $\mathbb{Y}^{\supset}$. The introduction of new tools such as *thick intervals* and *thick boxes* will allow us to propose an efficient algorithm that handles the uncertainty of sets in an elegant and efficient manner. Some elementary test-cases that cannot be handled easily and properly by existing methods show the efficiency of the approach.

Keywords: Set-Membership methods, Interval Analysis, Constraint programming, Uncertainty.

1. Introduction

Interval-based methods [1, 2] combined with constraint propagation [3, 4, 5] have been shown to be very efficient to deal with continuous constraint satisfaction problems (see, *e.g.*, [6, 7, 8]) and global optimization [9]. A specific and important constraint satisfaction problem is *set inversion* [10] which can also be interpreted as the inversion of a set-membership constraint [11]. Given a function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ and a set $\mathbb{Y} \subset \mathbb{R}^m$, set inversion aims at bracketing from inside and outside the set

$$\mathbb{X} = \mathbf{f}^{-1}(\mathbb{Y}). \tag{1}$$

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This formalism has been used for more than 20 years with interval methods to solve problems in bounded-error parameter estimation [12] , robot localization [13, 14, 15] and robust control [16, 17, 18]. Most interval algorithms for set-inversion alternate some interval tests or contractions [19] to certificate that a box (*i.e.*, a Cartesian product of intervals) is inside or outside the solution set $\mathbb{X}$ and bisect the boxes for which no conclusion can be reached.

In this paper, we consider the case where both $\mathbf{f}$ and $\mathbb{Y}$ are uncertain. A relaxation of the resulting uncertain constraints can be performed by adding quantifiers as made in [20, 21] for the linear case or by allowing a given number of constraints to be unsatisfied [22]. Here, we assume that $\mathbf{f} \in [\mathbf{f}] = [\mathbf{f}^-, \mathbf{f}^+]$ where $\mathbf{f}^-, \mathbf{f}^+$ are two known functions from $\mathbb{R}^n$ to $\mathbb{R}^m$. We also assume that the uncertain set $\mathbb{Y}$ satisfies $\mathbb{Y}^{\subset} \subset \mathbb{Y} \subset \mathbb{Y}^{\supset}$, where $\mathbb{Y}^{\subset}, \mathbb{Y}^{\supset}$ are two known subsets of $\mathbb{R}^m$. This amounts to saying that $\mathbb{Y}$ belongs to an interval of sets, denoted by $[\![\mathbb{Y}]\!]$, the lower and the upper bound of which are $\mathbb{Y}^{\subset}$ and $\mathbb{Y}^{\supset}$. We say that $[\mathbf{f}]$ is a *thick function* and that $[\![\mathbb{Y}]\!]$ is a *thick set* (also called *set interval* [23] [24]). Existing interval methods can still be used to deal with this type of uncertainties but they accumulate on a thick boundary which is called the *penumbra*. This accumulation makes classical interval methods inefficient, since they spend most of the computation time to test tiny boxes that are inside the penumbra.

Example 1. Consider the set inversion problem $\mathbb{X} = f^{-1}([y])$ with $[y] = [0, 4]$. We assume that f is uncertain and that we only know that for all $\mathbf{x}$

$$f(\mathbf{x}) \in [f](\mathbf{x}) = (x_1 - [a_1])^2 + (x_2 - [a_2])^2 . \tag{2}$$

with $a_1 \in [0, 1]$, $a_2 \in [0, 1]$. Note that f is not necessarily a circular paraboloid, and may correspond to any weird function satisfying the enclosure condition. Since, for all $\mathbf{x}$, $[f](\mathbf{x})$ is an interval of $\mathbb{R}$, the function $[f]$ is a thick function. More precisely, we have

$$[f](\mathbf{x}) = \left[f^-(\mathbf{x}), f^+(\mathbf{x})\right] \tag{3}$$

where

$$f^-(\mathbf{x}) = \min_{\mathbf{a} \in [0,1] \times [0,1]} (x_1 - a_1)^2 + (x_2 - a_2)^2 \tag{4}$$

and

$$f^+(\mathbf{x}) = \max_{\mathbf{a} \in [0,1] \times [0,1]} (x_1 - a_1)^2 + (x_2 - a_2)^2 . \tag{5}$$

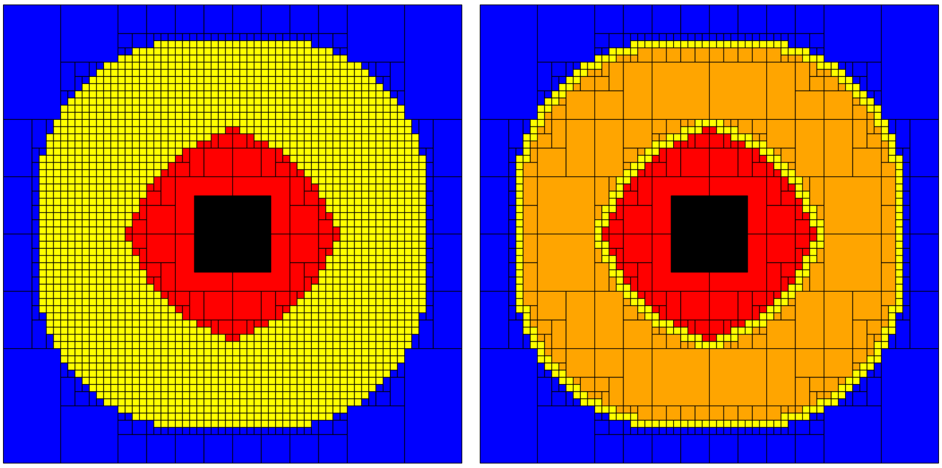

Figure 1: Left: Classical interval methods accumulate on the thick boundary (the penumbra). Right: the method we propose here will allow a fast treatment of the penumbra. The frame box is $[-2,4]\times[-2,4]$ and the black box corresponds to $[\mathbf{a}]$.

Using a classical interval arithmetic [1], we can easily test if a box $[\mathbf{x}]=[x_1]\times[x_2]$ is inside or outside the solution set:

$$\begin{array}{llll} (i) & ([x_1]-[a_1])^2+([x_2]-[a_2])^2\subset[y] & \Rightarrow & [\mathbf{x}]\subset\mathbb{X} \\ (ii) & ([x_1]-[a_1])^2+([x_2]-[a_2])^2\cap[y]=\emptyset & \Rightarrow & [\mathbf{x}]\cap\mathbb{X}=\emptyset. \end{array}$$

Now, we are not able conclude anything if none of these conditions is satisfied. Figure 1 (left) corresponds to the result of a paver based on these two tests (see the Set Inversion algorithm recalled at Subsection 4.1). Red boxes satisfy the inner test *(i)*, blue boxes satisfy the outer test (ii) and yellow boxes satisfy neither Test (i) nor Test (ii). The yellow boxes are not bisected by the paver since they reached the required accuracy. They cover a zone, called the *penumbra*, which corresponds to the part of the plane for which both the inner test and the outer test fail. Of course, if we were able to conclude that a box is inside the penumbra, many bisections would have been avoided. We would thus get a picture similar to Figure 1 (right) which is an approximation of a thick set with the inner part (red), the outer part (blue) and the penumbra (orange).

Now, when dealing with practical applications, the penumbra often exists as for instance when we want to characterize the zone that has actually been explored by a robot [25] or in case of partial observability [26]. Characterizing the penumbra from inside will

Subsets of $\mathbb{R}^n$:	$\mathbb{X}$
Intervals of $\mathbb{R}$:	$[a]$
Boxes of $\mathbb{R}^n$:	$[\mathbf{a}]$
Thick intervals:	$[\![a]\!] = [\![[a^-], [a^+]]\!]$
Thick boxes:	$[\![\mathbf{a}]\!] = [\![[\mathbf{a}^-], [\mathbf{a}^+]]\!]$
Thick sets:	$[\![\mathbb{A}]\!] = [\![\mathbb{A}^{\subset}, \mathbb{A}^{\supset}]\!]$
Thick functions:	$[f], [\mathbf{f}]$

Table 1: Notations for intervals, thick functions and thick sets

allow us to save computing time, but also to make the difference between the uncertainty due to the computation and that due to the initial uncertainties of the input parameters. The objective of this paper is to extend set inversion to the thick case (where a penumbra exists) and to show how to conclude that a box is inside the penumbra.

Notation. The notation to be used in this paper are given in Table 1. Vectors and vector-valued functions are written in bold font. For instance $a \in \mathbb{R}$, $\mathbf{b} = (b_1, b_2) \in \mathbb{R}^2$, $f(t) = sin(t)$, $\mathbf{g}(\mathbf{x}) = (x_1, x_1 + x_2)$. As often used in the interval community, intervals are denoted with brackets and sets are in *mathbb* font. But for thick sets, which are sets of subsets of $\mathbb{R}^n$, we will use the double brackets $[\![\ ]\!]$.

The paper is organized as follows. Section 2 proposes a formalization of the problem and introduces the concept of *thick set inversion*. Section 3 presents the new notion of thick intervals and thick boxes to be used for solving the thick set-inversion problem. Section 4 generalizes the classical set-inversion algorithm to the thick case by introducing the new notion of thick inclusion function. Section 5 illustrates the principle of the method on five test cases with one involving an actual underwater robot. Section 6 concludes the paper.

2. Problem statement

This section recalls some notions on lattices and intervals that will be used to formalize the problem of thick set inversion.

Lattices. Interval methods can be applied as soon as the set of domains for the variables has a lattice structure [27]. A *lattice* $(\mathcal{E}, \leq)$ is a partially ordered set, closed under least upper and greatest lower bounds [28]. The least upper bound of x and y is called the *join* and is denoted by $x \vee y$. The greatest lower bound is called the *meet* and is written as $x \wedge y$. Let us now give three examples.

- The set $(\mathbb{R}^n, \leq)$ is a lattice with respect to the partial order relation given by $\mathbf{x} \leq \mathbf{y} \Leftrightarrow \forall i \in \{1, \ldots, n\}, x_i \leq y_i$. We have $\mathbf{x} \wedge \mathbf{y} = (x_1 \wedge y_1, \ldots, x_n \wedge y_n)$ and $\mathbf{x} \vee \mathbf{y} = (x_1 \vee y_1, \ldots, x_n \vee y_n)$ where $x_i \wedge y_i = \min(x_i, y_i)$ and $x_i \vee y_i = \max(x_i, y_i)$.
- The set $(\mathbb{F}, \leq)$ of the functions which map $\mathbb{R}$ to $\mathbb{R}$ is a lattice with respect to the partial order relation given by $f \leq g \Leftrightarrow \forall t \in \mathbb{R}, f(t) \leq g(t)$. We have $f \wedge g : t \mapsto \min\{f(t), g(t)\}$ and $f \vee g : t \mapsto \max\{f(t), g(t)\}$.
- The set $\mathbb{IR}$ of closed intervals, as introduced by Moore [29], is a complete lattice with respect to the inclusion $\subset$. The meet corresponds to the intersection (generally denoted by $\cap$) and the join corresponds to the interval hull (generally denoted by $\sqcup$). For instance

$$[1,4] \cap [2,\infty] = [2,4] \text{ and } [1,4] \sqcup [8,9] = [1,9]. \tag{6}$$

A lattice $\mathcal{E}$ is *complete* if for all (finite or infinite) subsets $\mathcal{A}$ of $\mathcal{E}$, the least upper bound $\wedge\mathcal{A}$ and the greatest lower bound $\vee\mathcal{A}$ belong to $\mathcal{E}$. When a lattice $\mathcal{E}$ is not complete, We can add two elements corresponding to $\wedge\mathcal{A}$ and $\vee\mathcal{A}$ to make it complete. For instance, the set $\mathbb{R}$ is not a complete lattice whereas $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ is. By convention, for the empty set, we set $\wedge\emptyset = \vee\mathcal{E}$ and $\vee\emptyset = \wedge\mathcal{E}$.

Intervals. A *closed interval* (or *interval* for short) $[x]$ of a complete lattice $\mathcal{E}$ is a subset of $\mathcal{E}$ which satisfies $[x] = \{x \in \mathcal{E} \mid \wedge[x] \leq x \leq \vee[x]\}$. Both $\emptyset$ and $\mathcal{E}$ are intervals of $\mathcal{E}$. If we denote by $\mathbb{I}\mathcal{E}$ the set of all intervals of a complete lattice $(\mathcal{E}, \leq)$ then $(\mathbb{I}\mathcal{E}, \subset)$ is also a lattice. For two elements $[x] = [x^-, x^+]$ and $[y] = [y^-, y^+]$ of $\mathbb{I}\mathcal{E}$, we have:

$$\begin{array}{rcl} [x] \wedge [y] & = & [x^- \vee y^-, x^+ \wedge y^+] \\ [x] \vee [y] & = & [x^- \wedge y^-, x^+ \vee y^+]. \end{array} \tag{7}$$

The meet $[x] \wedge [y]$ is called the *intersection* and will denoted by $[x] \cap [y]$. The join $[x] \vee [y]$ is the *interval hull*, denoted by $[x] \sqcup [y]$. It should not be confused with the classical union $\cup$ of two intervals.

Remark 2. The bracket notation is here used to denote an interval. The brackets can be interpreted as an operator which associates to an unknown variable x, an interval domain $[x]$ which contains it. This operator is used when solving *Constraint Satisfaction Problems* with intervals [8]. When we define spontaneously an interval $[x]$, then, in the same time, we define implicitly a variable x which is enclosed by $[x]$.

Thick set. Denote by $(\mathcal{P}(\mathbb{R}^n), \subset)$ the powerset of $\mathbb{R}^n$ equipped with the inclusion $\subset$ as an order relation. $\mathcal{P}(\mathbb{R}^n)$ is a complete lattice with respect to $\subset$. The meet operator corresponds to the intersection and the join to the union. A *thick set* $[\![\mathbb{X}]\!]$ of $\mathbb{R}^n$ is an interval of $(\mathcal{P}(\mathbb{R}^n), \subset)$. If $[\![\mathbb{X}]\!]$ is a thick set of $\mathbb{R}^n$, there exist [24] two subsets of $\mathbb{R}^n$, called the *subset bound* $\mathbb{X}^{\subset}$ and the *supset bound* $\mathbb{X}^{\supset}$ such that

$$\begin{aligned} [\![\mathbb{X}]\!] &= [\![\mathbb{X}^{\subset}, \mathbb{X}^{\supset}]\!] \\ &= \{\mathbb{X} \in \mathcal{P}(\mathbb{R}^n) \mid \mathbb{X}^{\subset} \subset \mathbb{X} \subset \mathbb{X}^{\supset}\}\,. \end{aligned} \tag{8}$$

A thick set partitions $\mathbb{R}^n$ into three zones: the *clear zone* $\mathbb{X}^{\subset}$, the *penumbra* $\mathbb{X}^{\supset} \backslash \mathbb{X}^{\subset}$ and the *dark zone* $\mathbb{R}^n \backslash \mathbb{X}^{\supset}$. A thick set $[\![\mathbb{X}]\!]$ is a sub-lattice of $(\mathcal{P}(\mathbb{R}^n), \subset)$, *i.e.*, if $\mathbb{A} \in [\![\mathbb{X}]\!]$,$\mathbb{B} \in [\![\mathbb{X}]\!]$, then $\mathbb{A} \cap \mathbb{B} \in [\![\mathbb{X}]\!]$ and $\mathbb{A} \cup \mathbb{B} \in [\![\mathbb{X}]\!]$. The set of thick sets of $\mathbb{R}^n$ will be denoted by $\mathbb{I}\mathcal{P}(\mathbb{R}^n)$. If for the thick set $[\![\mathbb{X}]\!] = [\![\mathbb{X}^{\subset}, \mathbb{X}^{\supset}]\!]$, we have $\mathbb{X}^{\subset} = \mathbb{X}^{\supset}$ then $[\![\mathbb{X}]\!]$ is said to be *thin*. It corresponds to a singleton in $\mathcal{P}(\mathbb{R}^n)$ or equivalently a classical subset of $\mathbb{R}^n$.

For thick sets, we have two types of intersection:

$$\begin{aligned} [\![\mathbb{X}]\!] \cap [\![\mathbb{Y}]\!] &= \{\mathbb{Z} \in \mathcal{P}(\mathbb{R}^n) \,|\, \mathbb{Z} = \mathbb{X} \cap \mathbb{Y}, \mathbb{X} \in [\![\mathbb{X}]\!]\,, \mathbb{Y} \in [\![\mathbb{Y}]\!]\} \\ [\![\mathbb{X}]\!] \sqcap [\![\mathbb{Y}]\!] &= \{\mathbb{Z} \in \mathcal{P}(\mathbb{R}^n) \,|\, \mathbb{Z} \in [\![\mathbb{X}]\!]\,, \mathbb{Z} \in [\![\mathbb{Y}]\!]\}. \end{aligned} \tag{9}$$

The first $\cap$ corresponds to the extension to $\mathbb{I}\mathcal{P}(\mathbb{R}^n)$ of the intersection in $\mathcal{P}(\mathbb{R}^n)$ whereas the second $\sqcap$ corresponds to the intersection in $\mathbb{I}\mathcal{P}(\mathbb{R}^n)$. Therefore

$$\begin{aligned} \mathbb{X} \in [\![\mathbb{X}]\!]\,, \mathbb{Y} \in [\![\mathbb{Y}]\!] &\Rightarrow \mathbb{X} \cap \mathbb{Y} \in [\![\mathbb{X}]\!] \cap [\![\mathbb{Y}]\!] \\ \mathbb{Z} \in [\![\mathbb{X}]\!]\,, \mathbb{Z} \in [\![\mathbb{Y}]\!] &\Rightarrow \mathbb{Z} \in [\![\mathbb{X}]\!] \sqcap [\![\mathbb{Y}]\!]\,. \end{aligned} \tag{10}$$

As shown in [23], we have:

$$\begin{aligned} [\![\mathbb{X}]\!] \cap [\![\mathbb{Y}]\!] &= [\![\mathbb{X}^{\subset} \cap \mathbb{Y}^{\subset}, \mathbb{X}^{\supset} \cap \mathbb{Y}^{\supset}]\!] \\ [\![\mathbb{X}]\!] \sqcap [\![\mathbb{Y}]\!] &= [\![\mathbb{X}^{\subset} \cup \mathbb{Y}^{\subset}, \mathbb{X}^{\supset} \cap \mathbb{Y}^{\supset}]\!]. \end{aligned} \tag{11}$$

The same type of relations applies to the union.

Notation. We introduce a specific notation involving the quantifier $\forall$ when dealing with thick sets. Given two thick sets $[\![\mathbb{A}]\!]$ and $[\![\mathbb{B}]\!]$, we define:

$$\begin{array}{rcl} ([\![\mathbb{A}]\!] \subset [\![\mathbb{B}]\!])^{\forall} & \Leftrightarrow & \forall \mathbb{A} \in [\![\mathbb{A}]\!], \forall \mathbb{B} \in [\![\mathbb{B}]\!], \mathbb{A} \subset \mathbb{B} \\ ([\![\mathbb{A}]\!] \not\subset [\![\mathbb{B}]\!])^{\forall} & \Leftrightarrow & \forall \mathbb{A} \in [\![\mathbb{A}]\!], \forall \mathbb{B} \in [\![\mathbb{B}]\!], \mathbb{A} \not\subset \mathbb{B} \\ ([\![\mathbb{A}]\!] \cap [\![\mathbb{B}]\!] = \emptyset)^{\forall} & \Leftrightarrow & \forall \mathbb{A} \in [\![\mathbb{A}]\!], \forall \mathbb{B} \in [\![\mathbb{B}]\!], \mathbb{A} \cap \mathbb{B} = \emptyset \\ ([\![\mathbb{A}]\!] \cap [\![\mathbb{B}]\!] \neq \emptyset)^{\forall} & \Leftrightarrow & \forall \mathbb{A} \in [\![\mathbb{A}]\!], \forall \mathbb{B} \in [\![\mathbb{B}]\!], \mathbb{A} \cap \mathbb{B} \neq \emptyset. \end{array} \tag{12}$$

Thick function. Denote by $(\mathcal{F}(\mathbb{R}^n, \mathbb{R}^m), \leq)$ the set of all functions from $\mathbb{R}^n \to \mathbb{R}^m$ equipped with the order relation $\leq$ defined as follows

$$\mathbf{f} \leq \mathbf{g} \Leftrightarrow \forall i \in \{1, \ldots, m\}, \forall \mathbf{x} \in \mathbb{R}^n, f_i(\mathbf{x}) \leq g_i(\mathbf{x}). \tag{13}$$

The set $\mathcal{F}(\mathbb{R}^n, \mathbb{R}^m)$ is a lattice where the meet and the join are defined by

$$\mathbf{f} \wedge \mathbf{g}(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \wedge g_1(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \wedge g_m(\mathbf{x}) \end{pmatrix}, \tag{14}$$

and

$$\mathbf{f} \vee \mathbf{g}(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \vee g_1(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \vee g_m(\mathbf{x}) \end{pmatrix}. \tag{15}$$

A *thick function* $[\mathbf{f}]$ from $\mathbb{R}^n$ to $\mathbb{IR}^m$ is an interval of $(\mathcal{F}(\mathbb{R}^n, \mathbb{R}^m), \leq)$. For such a thick function $[\mathbf{f}]$, there exist two functions $\mathbf{f}^-$ and $\mathbf{f}^+$, called the *lower bound* and the *upper bound* such that

$$\begin{array}{rcl} [\mathbf{f}] & = & [\mathbf{f}^-, \mathbf{f}^+] \\ & = & \{\mathbf{f} \in \mathcal{F}(\mathbb{R}^n, \mathbb{R}^m) \mid \forall \mathbf{x} \in \mathbb{R}^n, \mathbf{f}^-(\mathbf{x}) \leq \mathbf{f}(\mathbf{x}) \leq \mathbf{f}^+(\mathbf{x})\}. \end{array}$$

A thick function $[\mathbf{f}]$ is a sublattice of $(\mathcal{F}(\mathbb{R}^n, \mathbb{R}^m), \leq)$, *i.e.*, if $\mathbf{f} \in [\mathbf{f}]$, $\mathbf{g} \in [\mathbf{f}]$, then $\mathbf{f} \wedge \mathbf{g} \in [\mathbf{f}]$ and $\mathbf{f} \vee \mathbf{g} \in [\mathbf{f}]$. Again, if $\mathbf{f}^- = \mathbf{f}^+$, $[\mathbf{f}]$ is said to be thin and corresponds to a singleton of $\mathcal{F}(\mathbb{R}^n, \mathbb{R}^m)$, or equivalently to a classical function from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Remark 3. The class of thick functions is not so restrictive. For instance, all set-valued functions of the form

$$F(\mathbf{x}) = \{f(\mathbf{x}, \mathbf{a}) \in \mathbb{R}, \mathbf{a} \in [\mathbf{a}] \subset \mathbb{R}^m\}, \tag{16}$$

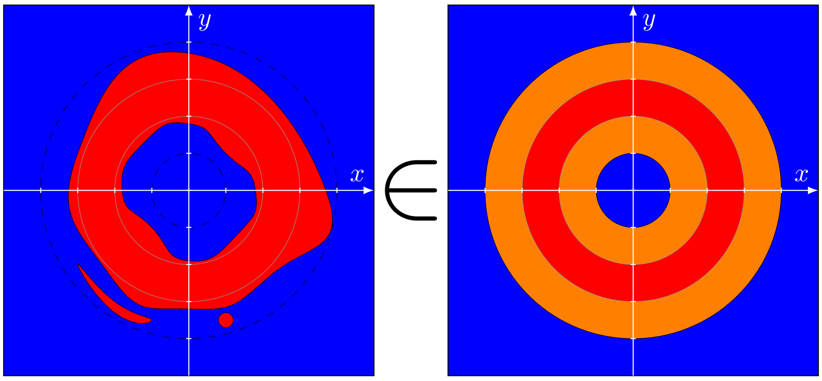

Figure 2: The thin set $\mathbb{X}$ (which is not a ring and even not connected) on the left belongs to the thick set $[\![\mathbb{X}]\!]$ on the right. $[\![\mathbb{X}]\!]$ contains all sets that enclose the red ring and which do not intersect the blue zone. .

where $[\mathbf{a}]$ is a box and where f is continuous with respect to $\mathbf{a}$, are thick functions. If now the box $[\mathbf{a}]$ is replaced by a disconnected set or when the function f is not scalar anymore, the function $F(\mathbf{x})$ has no reason to be a thick function.

Thick set inversion problem. A thick set inversion problem can be written as

$$\mathbb{X} = \mathbf{f}^{-1}(\mathbb{Y}),\, \mathbf{f} \in [\mathbf{f}] \text{ and } \mathbb{Y} \in [\![\mathbb{Y}]\!] \tag{17}$$

where $[\![\mathbb{Y}]\!]$ is a thick set and $[\mathbf{f}]$ is a thick function. The set $\mathbb{X}$ is said to be a *feasible solution* if

$$\exists \mathbf{f} \in [\mathbf{f}]\,, \exists \mathbb{Y} \in [\![\mathbb{Y}]\!]\,,\, \mathbb{X} = \mathbf{f}^{-1}(\mathbb{Y}). \tag{18}$$

The set of all feasible solutions is not a thick set in general as illustrated by the following example. Solving a thick set-inversion problem will amount to finding the smallest thick set which encloses all feasible solutions for (17).

Example 4. Let $\mathbf{f}(\mathbf{x}) = x_1^2 + x_2^2$ be a thin function from $\mathbb{R}^n$ to $\mathbb{R}$, and $\mathbb{Y}$ be a set such that $[2,3] \subset \mathbb{Y} \subset [1,4]$. If $\mathbb{X} = \mathbf{f}^{-1}(\mathbb{Y})$, we have:

$$\mathbf{f}^{-1}\left([2,3]\right) \subset \mathbb{X} \subset \mathbf{f}^{-1}\left([1,4]\right). \tag{19}$$

Now, all feasible $\mathbb{X}$ correspond to centered rings and it is clear that the inclusion (19) contains other types of sets as illustrated by Figure 2.

Theorem 5. *Given the thick function* $[\mathbf{f}]$ *and the thick set* $[\![\mathbb{Y}^{\subset},\mathbb{Y}^{\supset}]\!]$*, the smallest thick set which encloses all sets* $\mathbb{X}$ *such that*

$$\exists \mathbf{f} \in [\mathbf{f}]\,,\, \exists \mathbb{Y} \in [\![\mathbb{Y}]\!] \mid \mathbb{X} = \mathbf{f}^{-1}(\mathbb{Y}), \tag{20}$$

is the thick set $[\![\mathbb{X}^{\subset},\mathbb{X}^{\supset}]\!]$ *where*

$$\begin{aligned} \mathbb{X}^{\subset} &= \bigcap_{\mathbf{f}\in[\mathbf{f}]} \mathbf{f}^{-1}(\mathbb{Y}^{\subset}) \\ \mathbb{X}^{\supset} &= \bigcup_{\mathbf{f}\in[\mathbf{f}]} \mathbf{f}^{-1}(\mathbb{Y}^{\supset}). \end{aligned} \tag{21}$$

Proof. Denote by $\{\mathbb{X}_i\}_{i\in\mathbb{I}}$ the set of all solutions of (20). The smallest thick set $[\![\mathbb{X}]\!]$ containing $\{\mathbb{X}_i\}_{i\in\mathbb{I}}$ is the thick set

$$[\![\mathbb{X}]\!] = [\![\bigcap_{i\in\mathbb{I}}\{\mathbb{X}_i\},\bigcup_{i\in\mathbb{I}}\{\mathbb{X}_i\}]\!]. \tag{22}$$

Now, for each $\mathbb{X}_i \in \{\mathbb{X}_i\}_{i\in\mathbb{I}}$,

$$\exists \mathbf{f}_i \in [\mathbf{f}]\,,\, \exists \mathbb{Y}_i \in [\![\mathbb{Y}]\!] \mid \mathbb{X}_i = \mathbf{f}_i^{-1}(\mathbb{Y}_i). \tag{23}$$

Thus, $[\![\mathbb{X}]\!]$ is given by

$$\left[\!\!\left[\bigcap_{\mathbf{f}\in[\mathbf{f}]}\ \bigcap_{\mathbb{Y}\in[\![\mathbb{Y}]\!]} \mathbf{f}^{-1}(\mathbb{Y})\,,\ \bigcup_{\mathbf{f}\in[\mathbf{f}]}\ \bigcup_{\mathbb{Y}\in[\![\mathbb{Y}]\!]} \mathbf{f}^{-1}(\mathbb{Y})\right]\!\!\right]. \tag{24}$$

Now, since $\mathbf{f}^{-1}$ is monotonic with respect to the inclusion $\subset$, we get

$$[\![\mathbb{X}]\!] = \left[\!\!\left[\bigcap_{\mathbf{f}\in[\mathbf{f}]} \mathbf{f}^{-1}(\mathbb{Y}^{\subset}),\ \bigcup_{\mathbf{f}\in[\mathbf{f}]} \mathbf{f}^{-1}(\mathbb{Y}^{\supset})\right]\!\!\right].\blacksquare \tag{25}$$

Remark 6. Theorem 5 provides the exact formulation of the *thick set inversion* problem and defines the corresponding two sets we want to compute, *i.e.*, the two sets $\mathbb{X}^{\subset}$ and $\mathbb{X}^{\supset}$ as defined by (21). The main difficulty is to get an inner approximation of the penumbra $\mathbb{X}^{\supset}\backslash\mathbb{X}^{\subset}$. This justifies the introduction of the notion of thick intervals introduced in the following section.

3. Thick intervals

Denote by $\mathbb{IR}$ the set of all intervals of $\mathbb{R}$. A *thick interval* $[\![x]\!]$ (see, *e.g.*, [30]) is a subset of $\mathbb{IR}$ which can be written under the form

$$\begin{aligned} [\![x]\!] &= [\![[x^-]\,,[x^+]]\!] \\ &= \{[x^-,x^+] \in \mathbb{IR} \mid x^- \in [x^-] \text{ and } x^+ \in [x^+]\}\,. \end{aligned} \tag{26}$$

Here, $[x^-]$, $[x^+]$ are two intervals containing the *lower bound* $x^- \in \mathbb{R}$ and the *upper bound* $x^+ \in \mathbb{R}$ of an uncertain interval $[x^-, x^+]$. If we define the two intervals $[x^{\subset}] = [x^-] \cap [x^+]$ and $[x^{\supset}] = [x^-] \sqcup [x^+]$ of $\mathbb{R}$, called the *subset bound* and the *supset bound* of $[\![x]\!]$ then

$$[\![x]\!] \subset \{[x] \in \mathbb{IR} \mid [x^{\subset}] \subset [x] \subset [x^{\supset}]\}\,, \tag{27}$$

with an equality if $[x^-] \cap [x^+] \neq \emptyset$. As a consequence, a thick interval is not necessarily a thick set: it is more precise or equivalently it is narrower. This can be explained using the *endpoints diagram* [31] (see Figure 3) where an interval is seen as a point of $\mathbb{R}^2$. For instance, to the interval $[1, 7]$, we associate the point with coordinates $(1, 7)$. The degenerated intervals, such as $[2, 2]$, all belong to the diagonal. This representation provides a geometrical representation of the relation between intervals. For instance, $[x] \subset [y]$ if $[y]$ is at the top left of $[x]$. The intersection between two intervals (or the interval hull) is obtained by taking the bottom-right corner (or the top left corner) of the smallest box which encloses the two interval points. For instance, if $[x] = [1, 4]$ and $[y] = [2, 5]$, the enclosing interval box is painted red. The top left interval is $[x] \cup [y] = [1, 5]$ and the bottom-right interval is $[x] \cap [y] = [2, 4]$. This red box corresponds to the thick box $[\![a]\!] = [\![[1, 2], [4, 5]]\!]$. The subset and supset bounds are $[a^{\subset}] = [2, 4]$ and $[a^{\supset}] = [1, 5]$.

In this figure, the orange polygon corresponds to the thick interval $[\![b]\!] = [\![[3, 7], [6, 8]]\!]$. The subset and supset bounds are $[b^{\subset}] = \emptyset$ and $[b^{\supset}] = [3, 8]$. As illustrated by the gray zone of Figure 3 (left), a subset-supset representation adds pessimism when the subset bound is empty, *i.e.*, a subset-supset representation may contain more intervals. For instance, if $[b] = [4, 5]$, we have $\emptyset \subset [b] \subset [3, 8]$, but $[b] \notin [\![[3, 7], [6, 8]]\!]$. The corresponding lower and upper interval bounds are represented on Figure 3 (right). Note that when the subset bound is not empty (as for the thick box $[\![\mathbf{a}]\!]$ in red), no pessimism is added and both representations are equivalent.

Due to this pessimism, we will prefer to use a representation based on lower-upper bounds instead of the notation based on the subset-supset bounds. As already seen for thick sets (see (11)), set membership operations such as the union or the intersection can easily be extended to thick intervals. An extension for all classical operators of interval arithmetic is also valid. More precisely, if $\diamond \in \{+, -, \cdot, \cap, \sqcup, \dots\}$, we define

$$[\![x]\!] \diamond [\![y]\!] \quad = \quad \{[x] \diamond [y] \mid [x] \in [\![x]\!]\,, [y] \in [\![y]\!]\}. \tag{28}$$

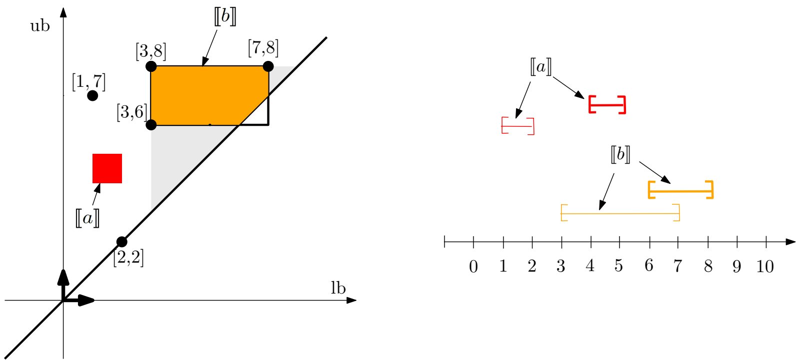

Figure 3: In the endpoints diagram, an interval is represented by a point (here the small black disks)

For instance, if $[\![a]\!] = [\![[1,2],[4,5]]\!]$ and $[\![b]\!] = [\![[3,7],[6,8]]\!]$, we have

$$\begin{aligned}
[\![a]\!] + [\![b]\!] &= [\![[1,2]+[3,7],[4,5]+[6,8]]\!] \\
&= [\![[4,9],[10,13]]\!] \\
[\![a]\!] \cap [\![b]\!] &= [\![max([1,2],[3,7]), min([4,5],[6,8])]\!] \\
&= [\![[3,7],[4,5]]\!] \\
[\![a]\!] \sqcup [\![b]\!] &= [\![min([1,2],[3,7]), max([4,5],[6,8])]\!] \\
&= [\![[1,2],[6,8]]\!] \,.
\end{aligned}$$

An interpretation for these formula is the following: if $[a] \in [\![a]\!]$ and $[b] \in [\![b]\!]$ then the intervals $[x] = [a]+[b]$, $[y] = [a] \cap [b]$, $[z] = [a] \sqcup [b]$ satisfy $[x] \in [\![a]\!] + [\![b]\!]$,$[y] \in [\![a]\!] \cap [\![b]\!]$, $[z] \in [\![a]\!] \sqcup [\![b]\!]$.

The following proposition shows how to compare, from a practical point of vue, two thick intervals.

Proposition 7. *Given two thick intervals* $[\![a]\!] = [\![[a^-],[a^+]]\!]$ *and* $[\![b]\!] = [\![[b^-],[b^+]]\!]$*, we have*

$$\begin{array}{lll}
(i) & ([\![a]\!] \subset [\![b]\!])^{\forall} \Leftrightarrow & \begin{cases} [b]^- - [a]^- \subset \mathbb{R}^- & \wedge \\ [a]^+ - [b]^+ \subset \mathbb{R}^- \end{cases} \\
(ii) & ([\![a]\!] \not\subset [\![b]\!])^{\forall} \Leftrightarrow & \begin{cases} [a]^- - [b]^- \subset \mathbb{R}^- & \vee \\ [b]^+ - [a]^+ \subset \mathbb{R}^- \end{cases} \\
(iii) & ([\![a]\!] \cap [\![b]\!] = \emptyset)^{\forall} \Leftrightarrow & \begin{cases} [a]^+ - [b]^- \subset \mathbb{R}^- & \vee \\ [b]^+ - [a]^- \subset \mathbb{R}^- \end{cases} \\
(iv) & ([\![a]\!] \cap [\![b]\!] \neq \emptyset)^{\forall} \Leftrightarrow & \begin{cases} [b]^- - [a]^+ \subset \mathbb{R}^- & \wedge \\ [a]^- - [b]^+ \subset \mathbb{R}^- \end{cases}
\end{array}$$

PROOF. (i) Consider two intervals $[a]$ and $[b]$ of $\mathbb{R}$. (i) The inclusion $[a] \subset [b]$ is satisfied iff

$$b^- \leq a^- \text{ and } a^+ \leq b^+. \tag{29}$$

Thus, the inclusion is true for all $[a] \in [\![a]\!]$ and all $[b] \in [\![b]\!]$ iff (i) is satisfied.

(ii) We have $[a] \not\subset [b]$ iff $b^- > a^-$ or $a^+ > b^+$. Thus, the inclusion is unsatisfied for all $[a] \in [\![a]\!]$ and all $[b] \in [\![b]\!]$ iff (ii) is satisfied.

(iii) The two intervals $[a]$ and $[b]$ are disjoint iff $b^- > a^+$ or $a^- > b^+$. Therefore, they are disjoint for all $[a] \in [\![a]\!]$ and all $[b] \in [\![b]\!]$ iff (iii) is true.

(iv) The two intervals $[a]$ and $[b]$ overlap iff $b^- \leq a^+$ and $a^- \leq b^+$. Thus, they overlap for all $[a] \in [\![a]\!]$ and all $[b] \in [\![b]\!]$ iff (iv) is true.

Example 8. The two intervals $[a] = [1,5]$ and $[b] \in [\![b]\!] = [\![[2,4],[3,6]]\!]$, depicted in Figure 4, overlap for all feasible $[b]$, *i.e.*, $([a] \cap [\![b]\!] \neq \emptyset)^{\forall}$. This can be checked using Proposition 7, iv:

$$[2,4] - 5 \subset \mathbb{R}^- \text{ and } 1 - [3,6] \subset \mathbb{R}^-. \tag{30}$$

Note that using the subset-supset bounds, we could not reach this conclusion. Indeed, the subset-supset approximation of $[\![b]\!]$ is

$$\emptyset \subset [b] \subset [2,6]. \tag{31}$$

The interval $[b] = [6,6]$ is consistent with this inclusion and does not intersect $[a]$. The green zone represents $[a] \cap [\![b]\!]$, the set of all intervals $[a] \cap [b]$ such that $[b] \in [\![b]\!]$.

Thick boxes. Denote by $\mathbb{IR}^n$ the set of all boxes of $\mathbb{R}^n$. A *thick box* $[\![\mathbf{x}]\!]$ is a set of boxes of $\mathbb{IR}^n$ which can be defined as

$$[\![\mathbf{x}]\!] = \left\{ \left[\mathbf{x}^-, \mathbf{x}^+\right] \in \mathbb{IR}^n \mid \mathbf{x}^- \in \left[\mathbf{x}^-\right], \mathbf{x}^+ \in \left[\mathbf{x}^+\right] \right\} \tag{32}$$

where $[\mathbf{x}^-]$ and $[\mathbf{x}^+]$ are boxes of $\mathbb{R}^n$. The set of thick boxes of $\mathbb{R}^n$ is denoted by $\mathbb{IIR}^n$. A thick box can be seen as an interval of boxes, *i.e.*, an interval of intervals of $\mathbb{R}^n$. This is illustrated by Figure 5 which shows four thin boxes all contained in the thick box $[\![\mathbf{x}]\!] = [\![[\mathbf{x}^-], [\mathbf{x}^+]]\!]$. Since the two box bounds of $[\mathbf{x}^-]$ and $[\mathbf{x}^+]$ are boxes of $\mathbb{R}^n$, we could decompose them as the Cartesian product of n intervals:

$$\begin{array}{rcl} [\mathbf{x}^-] & = & [x_1^-] \times \cdots \times [x_n^-] \\ [\mathbf{x}^+] & = & [x_1^+] \times \cdots \times [x_n^+]. \end{array} \tag{33}$$

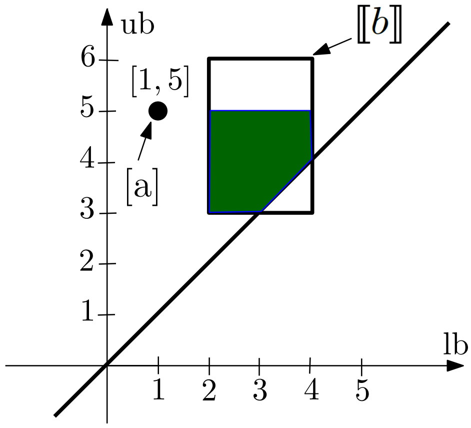

Figure 4: A subset-supset approximation of $[\![b]\!]$ adds pessimism and thus fails to conclude that $[a]$ and $[b]$ always overlap. The green zone corresponds to the thick interval $[a] \cap [\![b]\!]$.

We define the ith component $[\![x_i]\!]$ of the thick box $[\![[\mathbf{x}^-], [\mathbf{x}^+]]\!]$ as the thick interval $[\![x_i]\!] = [\![[x_i^-], [x_i^+]]\!]$.

The following proposition will allow us to compare two thick boxes, with respect to the inclusion, from their interval components.

Proposition 9. *Given two thick boxes* $[\![\mathbf{a}]\!]=[\![[\mathbf{a}^-], [\mathbf{a}^+]]\!]$ *and* $[\![\mathbf{b}]\!]=[\![[\mathbf{b}^-], [\mathbf{b}^+]]\!]$ *of* $\mathbb{R}^n$*, we have*

$$\begin{array}{ll}(i) & ([\![\mathbf{a}]\!] \subset [\![\mathbf{b}]\!])^{\forall} \Leftrightarrow \forall i \in \{1,\dots,n\}, ([\![a_i]\!] \subset [\![b_i]\!])^{\forall} \\ (ii) & ([\![\mathbf{a}]\!] \not\subset [\![\mathbf{b}]\!])^{\forall} \Leftrightarrow \exists i \in \{1,\dots,n\}, ([\![a_i]\!] \not\subset [\![b_i]\!])^{\forall} \\ (iii) & ([\![\mathbf{a}]\!] \cap [\![\mathbf{b}]\!] = \emptyset)^{\forall} \Leftrightarrow \exists i \in \{1,\dots,n\}, ([\![a_i]\!] \cap [\![b_i]\!] = \emptyset)^{\forall} \\ (iv) & ([\![\mathbf{a}]\!] \cap [\![\mathbf{b}]\!] \neq \emptyset)^{\forall} \Leftrightarrow \forall i \in \{1,\dots,n\}, ([\![a_i]\!] \cap [\![b_i]\!] \neq \emptyset)^{\forall}\end{array} \quad (34)$$

PROOF. This proof is a direct consequence of Proposition 7 and of the fact that

$$\begin{array}{rcl}[\mathbf{a}] \subset [\mathbf{b}] & \Leftrightarrow & \forall i, [a_i] \subset [b_i] \\ [\mathbf{a}] \not\subset [\mathbf{b}] & \Leftrightarrow & \exists i, [a_i] \not\subset [b_i] \\ [\mathbf{a}] \cap [\mathbf{b}] = \emptyset & \Leftrightarrow & \exists i, [a_i] \cap [b_i] = \emptyset \\ [\mathbf{a}] \cap [\mathbf{b}] \neq \emptyset & \Leftrightarrow & \forall i, [a_i] \cap [b_i] \neq \emptyset. \blacksquare\end{array} \quad (35)$$

4. Thick set inversion

This section generalizes to set inversion algorithm [32] to the thick case, as defined by Theorem 5.

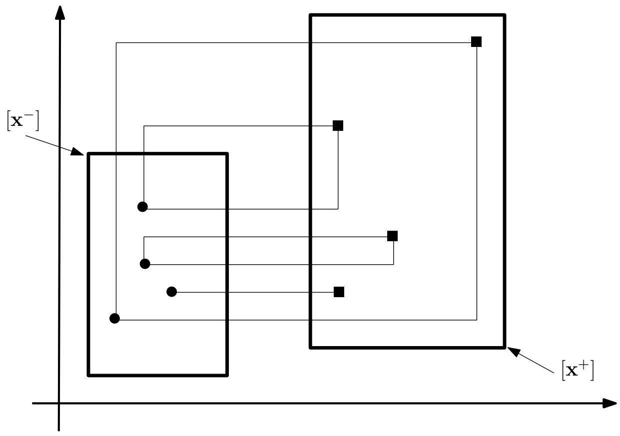

Figure 5: The four boxes (thin) all belong to the thick box $[\![\mathbf{x}]\!] = [\![[\mathbf{x}^-], [\mathbf{x}^+]]\!]$

4.1. Set inversion

Given a function $\mathbf{f}$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ and a (thin) set $\mathbb{Y} \subset \mathbb{R}^n$, solving the set inversion problem, denoted by $\mathbb{X} = \mathbf{f}^{-1}(\mathbb{Y})$, is classically performed using an *inclusion function* $[\mathbf{f}] : \mathbb{IR}^n \to \mathbb{IR}^m$ of $\mathbf{f}$ [1], *i.e.*, an interval function such that

$$\mathbf{a} \in [\mathbf{x}] \Rightarrow \mathbf{f}(\mathbf{a}) \in [\mathbf{f}]([\mathbf{x}]). \tag{36}$$

Most algorithms for set-inversion decompose $\mathbb{R}^n$ into boxes [33][32]. If a given box $[\mathbf{x}]$ satisfies $[\mathbf{f}]([\mathbf{x}]) \subset \mathbb{Y}$ then it is proved to be inside the solution set $\mathbb{X}$. If $[\mathbf{f}]([\mathbf{x}]) \cap \mathbb{Y} = \emptyset$ then it is proved to be outside $\mathbb{X}$. If it satisfies none of the previous tests, it is bisected until it becomes too small. A possible implementation for set inversion is given by Algorithm 1 which is called Sivia (Set Inversion Via Interval Analysis) [32].

When the algorithm terminates, we have [32]

$$\bigcup \mathcal{L}^{clear} \subset \mathbb{X}$$
$$\bigcup \mathcal{L}^{dark} \cap \mathbb{X} = \emptyset.$$

For the thick case, we have a thick function $[\mathbf{f}]$ from $\mathbb{R}^n$ to $\mathbb{IR}^m$ and a thick set $[\![\mathbb{Y}]\!] \in \mathbb{IP}(\mathbb{R}^n)$. We want to compute an approximation of the set of all sets $\mathbb{X} = \mathbf{f}^{-1}(\mathbb{Y})$, assuming that $\mathbf{f} \in [\mathbf{f}]$ and $\mathbb{Y} \in [\![\mathbb{Y}]\!]$. This problem, formalized by Theorem 5, is called a *thick set inversion problem*, denoted by

$$[\![\mathbb{X}]\!] = [\mathbf{f}]^{-1}([\![\mathbb{Y}]\!]). \tag{37}$$

Algorithm 1 Set inversion algorithm: SIVIA

Input: $[\mathbf{x}]$, ε, $\mathbb{Y}$, $[\![\mathbf{f}]\!]$

Output: $\mathcal{L}^{clear}$, $\mathcal{L}^{dark}$

```
L = {[x]}, L^clear = ∅, L^dark = ∅
while L ≠ ∅ do
    unstack L into [x]
    if ([f]([x]) ⊂ Y) then
        push [x] into L^clear
    else if ([f]([x]) ∩ Y = ∅) then
        push [x] into L^dark
    else if width([x]) > ε then
        bisect [x] perpendicularly to its largest side and push the two resulting boxes in L
    end if
end while
```

We propose to compute an approximation of $[\![\mathbb{X}]\!]$ by decomposing $\mathbb{R}^n$ into three subsets: the clear zone $\mathbb{X}^{\subset}$, the penumbra $\mathbb{X}^{\supset}\backslash\mathbb{X}^{\subset}$ and the dark zone $\mathbb{R}^n\backslash\mathbb{X}^{\supset}$. In our approach, a paver performs the decomposition of $\mathbb{R}^n$ into boxes and a thick extension of an inclusion function is used to classify boxes.

4.2. Thick inclusion function

The function $[\![\mathbf{f}]\!] : \mathbb{IR}^n \to \mathbb{IIR}^m$ is a *thick inclusion function* of the thick function $[\mathbf{f}] : \mathbb{R}^n \to \mathbb{IR}^m$ if

$$\mathbf{a} \in [\mathbf{x}] \Rightarrow [\mathbf{f}]\,(\mathbf{a}) \in [\![\mathbf{f}]\!]\,([\mathbf{x}])\,. \tag{38}$$

Theorem 10. *Consider a thick function $[\mathbf{f}]\,(\mathbf{x})$ and denote by $[\mathbf{f}^-], [\mathbf{f}^+]$ two inclusion functions for the bounds $\mathbf{f}^-, \mathbf{f}^+$ of $[\mathbf{f}]$. The function $[\![\mathbf{f}]\!] : \mathbb{IR}^n \to \mathbb{IIR}^m$ defined by*

$$[\![\mathbf{f}]\!]\,([\mathbf{x}]) = \left[\!\left[\left[\mathbf{f}^-\right]([\mathbf{x}])\,, \left[\mathbf{f}^+\right]([\mathbf{x}]) \right]\!\right], \tag{39}$$

is a thick inclusion function for $[\mathbf{f}]\,(\mathbf{x})$.

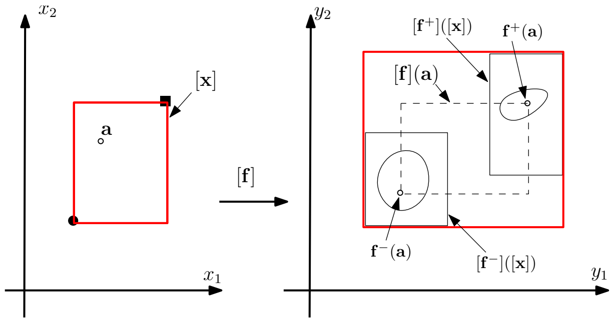

Figure 6: The thick inclusion function $[\![\mathbf{f}]\!]([\mathbf{x}])$ encloses all boxes $[\mathbf{f}](\mathbf{a})$ where $\mathbf{a} \in [\mathbf{x}]$.

Proof. Since $[\mathbf{f}^-], [\mathbf{f}^+]$ are two inclusion functions for $\mathbf{f}^-, \mathbf{f}^+$, we have

$$\mathbf{a} \in [\mathbf{x}] \Rightarrow \begin{cases} \mathbf{f}^-(\mathbf{a}) & \in & [\mathbf{f}^-]([\mathbf{x}]) \\ \mathbf{f}^+(\mathbf{a}) & \in & [\mathbf{f}^+]([\mathbf{x}]) \end{cases} \tag{40}$$

Now, the right hand side is equivalent to $[\mathbf{f}](\mathbf{a}) \in [\![\mathbf{f}]\!]([\mathbf{x}])$. ■

As illustrated by Figure 6, the thick box $[\![\mathbf{f}]\!]([\mathbf{x}])$ encloses the set of all boxes $[\mathbf{f}](\mathbf{a})$ with $\mathbf{a} \in [\mathbf{x}]$. The vector $\mathbf{a} \in [\mathbf{x}] \subset \mathbb{R}^2$ has an image $[\mathbf{f}](\mathbf{a})$ which is a box of $\mathbb{R}^2$ with a lower bound $\mathbf{f}^-(\mathbf{a})$ and an upper bound $\mathbf{f}^+(\mathbf{a})$. Using a classical interval arithmetic, we are able to get inclusion functions $[\mathbf{f}^-]$ and $[\mathbf{f}^+]$ for $\mathbf{f}^-$ and $\mathbf{f}^+$. The boxes $[\mathbf{f}^-]([\mathbf{x}])$ and $[\mathbf{f}^+]([\mathbf{x}])$ contain $\mathbf{f}^-(\mathbf{a})$ and $\mathbf{f}^+(\mathbf{a})$. Therefore, the box $[\mathbf{f}](\mathbf{a})$ is inside the thick box $[\![\mathbf{f}]\!]([\mathbf{x}]) = [\![[\mathbf{f}^-]([\mathbf{x}]), [\mathbf{f}^+]([\mathbf{x}])]\!]$.

4.3. Algorithm

Algorithm 2, named ThickSivia (Thick Set Inversion Via Interval Analysis), provides an approximation of the solution of the thick set inversion problem $[\![\mathbb{X}]\!] = [\mathbf{f}]^{-1}([\![\mathbb{Y}]\!])$. The input of this algorithm are (1) the box $[\mathbf{x}]$ which is assumed to be large enough to contains $\mathbb{X}^{\supset}$, the upper bound of $[\![\mathbb{X}]\!]$, (2) an accuracy $\varepsilon > 0$, (3) the thick inclusion function $[\![\mathbf{f}]\!]$, and (4) thick set $[\![\mathbb{Y}]\!]$. The output is an approximation of the thick set $[\![\mathbb{X}]\!] = [\![\mathbb{X}^{\subset}, \mathbb{X}^{\supset}]\!]$. The algorithm decomposes the initial box $[\mathbf{x}]$ into four non-overlapping subpavings: (1) The inner subpaving $\mathcal{L}^{clear}$ which contains boxes that have been proved to be inside the clear zone $\mathbb{X}^{\subset}$, (2) the outer subpaving $\mathcal{L}^{dark}$ which contains boxes that have been proved to be outside $\mathbb{X}^{\supset}$ (*i.e.*, inside the dark zone), (3) the subpaving

$\mathcal{L}^{penumbra}$ which contains boxes that have been proved to be inside the penumbra $\mathbb{X}^{\supset}\backslash\mathbb{X}^{\subset}$ and (4) the subpaving made with boxes that have been rejected (for which nothing is known and with a width smaller than the desired level of precision ε).

Algorithm 2 Thick set inversion algorithm: THICKSIVIA

Input: $[\mathbf{x}]$, ε, $[\![\mathbb{Y}]\!]$, $[\![\mathbf{f}]\!]$

Output: $\mathcal{L}^{clear}$, $\mathcal{L}^{penumbra}$, $\mathcal{L}^{dark}$

$\mathcal{L} = \{[\mathbf{x}]\}$, $\mathcal{L}^{clear} = \emptyset$, $\mathcal{L}^{penumbra} = \emptyset$, $\mathcal{L}^{dark} = \emptyset$

while $\mathcal{L} \neq \emptyset$ **do**

  Unstack $\mathcal{L}$ into $[\mathbf{x}]$

  if $([\![\mathbf{f}]\!]([\mathbf{x}]) \subset \mathbb{Y}^{\subset})^{\forall}$ **then**

   push $[\mathbf{x}]$ into $\mathcal{L}^{clear}$

  else if $([\![\mathbf{f}]\!]([\mathbf{x}]) \cap \mathbb{Y}^{\supset} = \emptyset)^{\forall}$ **then**

   push $[\mathbf{x}]$ into $\mathcal{L}^{dark}$

  else if $([\![\mathbf{f}]\!]([\mathbf{x}]) \cap (\mathbb{Y}^{\supset}\backslash\mathbb{Y}^{\subset}) \neq \emptyset)^{\forall}$ **then**

   push $[\mathbf{x}]$ into $\mathcal{L}^{penumbra}$

  else if width$([\mathbf{x}]) > \varepsilon$ **then**

   bisect $[\mathbf{x}]$ perpendicularly to its largest side and push the two resulting boxes in $\mathcal{L}$

  end if

end while

Remark 11. In the algorithm, we have several tests on thick boxes that require comparisons on thick boxes as introduced in Section 3. Consider for instance the test $([\![\mathbf{f}]\!]([\mathbf{x}]) \cap \mathbb{Y}^{\supset} = \emptyset)^{\forall}$ and assume that $\mathbb{Y}^{\supset}$ is made with boxes $\{[\mathbf{y}](1), \ldots, [\mathbf{y}](\bar{k})\}$ (this will be the case in the applications presented in Section 5). Since we have

$$\left([\![\mathbf{f}]\!]([\mathbf{x}]) \cap \mathbb{Y}^{\supset} = \emptyset\right)^{\forall} \Leftrightarrow \forall k, ([\![\mathbf{f}]\!]([\mathbf{x}]) \cap [\mathbf{y}](k) = \emptyset)^{\forall}.$$

Our test amounts to checking that $([\![\mathbf{a}]\!] \cap [\mathbf{b}] = \emptyset)^{\forall}$ where $[\![\mathbf{a}]\!] = [\![\mathbf{f}]\!]([\mathbf{x}])$ and $[\mathbf{b}] = [\mathbf{y}](k)$.

Therefore

$$\begin{aligned} & (\llbracket \mathbf{a} \rrbracket \cap [\mathbf{b}] = \emptyset)^{\forall} \\ \Leftrightarrow\quad & \exists i \in \{1,\dots,n\}, (\llbracket a_i \rrbracket \cap [b_i] \neq \emptyset)^{\forall} && \text{(see Proposition 9)} \\ \Leftrightarrow\quad & \exists i \in \{1,\dots,n\}, b_i^- - [a_i]^+ \subset \mathbb{R}^- \wedge [a_i]^- - b_i^+ \subset \mathbb{R}^- && \text{(see Proposition 7)} \\ \Leftrightarrow\quad & \exists i \in \{1,\dots,n\},\ b_i^- - lb\left([a_i]^+\right) \leq 0 \wedge (ub\left([a_i]^-\right)) - b_i^+ \leq 0 \end{aligned}$$

where $lb([a_i]^+)$ is the lower bound of the interval $[a_i]^+$ and $ub([a_i]^-)$ is the upper bound of $[a_i]^-$. We thus get Algorithm 3. The same type of algorithm applies to test if $(\llbracket \mathbf{f} \rrbracket ([\mathbf{x}]) \subset \mathbb{Y}^{\subset})^{\forall}$ or $(\llbracket \mathbf{f} \rrbracket ([\mathbf{x}]) \cap (\mathbb{Y}^{\supset} \backslash \mathbb{Y}^{\subset}) \neq \emptyset)^{\forall}$.

Algorithm 3 Test if $(\llbracket \mathbf{f} \rrbracket ([\mathbf{x}]) \cap \mathbb{Y}^{\supset} = \emptyset)^{\forall}$

Input: $\llbracket \mathbf{a} \rrbracket = \llbracket \mathbf{f} \rrbracket ([\mathbf{x}]), \mathbb{Y}^{\supset} = \{[\mathbf{y}](1), \dots, [\mathbf{y}](\bar{k})\}$

```
1: for k = 1 to k̄ do
       [b] = [y](k)
2:     for i = 1 to n do
       If  b_i^- − lb([a_i]^+) > 0  ∨  (ub([a_i]^-)) − b_i^+ > 0  then Return False. End.
3:     end for
4: end for
5: Return True
```

4.4. Properties

Termination

The algorithm always terminates in less than $\lambda = \left(\frac{\text{with}([\mathbf{x}])}{\varepsilon}\right)^{dim(\mathbf{x})}$ iterations, where $[\mathbf{x}]$ it the input box. The number λ is huge and corresponds to the worst case situation where all tests fail and the algorithm returns $\mathbb{X}^{clear} = \emptyset$, $\mathbb{X}^{penumbra} = \emptyset$, $\mathbb{X}^{dark} = \emptyset$. In practice, as shown in [34] the number of iterations is $O\left(A\left(\frac{1}{\varepsilon}\right)^{dim(\mathbf{x})-1}\right)$, where A is the area of the accumulation zone which is composed here with the boundary of the penumbra.

Enclosure

The algorithm computes guaranteed inner and outer approximations of the solution set of the thick set inversion problem. This is asserted by the following theorem.

Theorem 12. *The algorithm returns an approximation of the thick set inversion problem* $[\![\mathbb{X}^{\subset},\mathbb{X}^{\supset}]\!] = [\mathbf{f}]^{-1}([\![\mathbb{Y}]\!])$ *under the form of 3 subpavings (i.e., union of boxes):* $\mathcal{L}^{clear}$, $\mathcal{L}^{penumbra}$, $\mathcal{L}^{dark}$. *This approximation satisfies*

$$\begin{array}{ll} (i) & \bigcup \mathcal{L}^{clear} \subset \mathbb{X}^{\subset} \\ (ii) & \bigcup \mathcal{L}^{penumbra} \subset \mathbb{X}^{\supset}\backslash\mathbb{X}^{\subset} \\ (iii) & \bigcup \mathcal{L}^{dark} \cap \mathbb{X}^{\supset} = \emptyset. \end{array}$$

Proof. To prove the Theorem, we need to show that, for a box $[\mathbf{x}]$, we have (see Figure 7):

$$\begin{array}{llll} (i) & ([\![\mathbf{f}]\!]([\mathbf{x}]) \subset \mathbb{Y}^{\subset})^{\forall} & \Rightarrow & [\mathbf{x}] \subset \mathbb{X}^{\subset} \\ (ii) & ([\![\mathbf{f}]\!]([\mathbf{x}]) \cap \mathbb{Y}^{\supset} = \emptyset)^{\forall} & \Rightarrow & [\mathbf{x}] \cap \mathbb{X}^{\supset} = \emptyset \\ (iii) & ([\![\mathbf{f}]\!]([\mathbf{x}]) \cap (\mathbb{Y}^{\supset}\backslash\mathbb{Y}^{\subset}) \neq \emptyset)^{\forall} & \Rightarrow & [\mathbf{x}] \subset \mathbb{X}^{\supset}\backslash\mathbb{X}^{\subset}. \end{array} \quad (41)$$

Let us first prove (i). The left hand side of (i) implies that

$$\forall [\mathbf{a}] \in [\![\mathbf{f}]\!]([\mathbf{x}]),\ [\mathbf{a}] \subset \mathbb{Y}^{\subset}. \quad (42)$$

Take $\mathbf{x} \in [\mathbf{x}]$, and we show that $\mathbf{x} \in \mathbb{X}^{\subset}$. Since $\mathbf{x} \in [\mathbf{x}]$, we have $[\![\mathbf{f}]\!](\mathbf{x}) \subset [\![\mathbf{f}]\!]([\mathbf{x}])$ and the previous formula implies.

$$\forall [\mathbf{a}] \in [\![\mathbf{f}]\!](\mathbf{x}),\ [\mathbf{a}] \subset \mathbb{Y}^{\subset}. \quad (43)$$

Now, $[\![\mathbf{f}]\!](\mathbf{x})$ is a singleton in $\mathbb{IR}^n$ which contains the single box $[\mathbf{f}](\mathbf{x})$. Thus, (43) becomes $[\mathbf{f}](\mathbf{x}) \subset \mathbb{Y}^{\subset}$, which implies

$$\forall \mathbf{f} \in [\mathbf{f}],\ \mathbf{f}(\mathbf{x}) \in \mathbb{Y}^{\subset} \quad (44)$$

or equivalently $\forall \mathbf{f} \in [\mathbf{f}],\ \mathbf{x} \in \mathbf{f}^{-1}(\mathbb{Y}^{\subset})$. We get

$$\mathbf{x} \in \bigcap_{\mathbf{f}\in[\mathbf{f}]} \mathbf{f}^{-1}(\mathbb{Y}^{\subset}) \overset{(21)}{=} \mathbb{X}^{\subset}. \quad (45)$$

The same reasoning applies to prove (ii). For (iii), assume that the left hand side of (iii) is satisfied. Take one $\mathbf{x} \in [\mathbf{x}]$, the quantity $[\![\mathbf{f}]\!]([\mathbf{x}])$ becomes a singleton in $\mathbb{IR}^n$, *i.e.*, a box of $\mathbb{R}^n$. We have

$$\begin{array}{ll} & ([\mathbf{f}](\mathbf{x}) \cap (\mathbb{Y}^{\supset}\backslash\mathbb{Y}^{\subset}) \neq \emptyset) \\ \Leftrightarrow & \exists \mathbf{f} \in [\mathbf{f}], \mathbf{x} \in \mathbf{f}^{-1}(\mathbb{Y}^{\supset}\backslash\mathbb{Y}^{\subset}) \\ \Leftrightarrow & \exists \mathbf{f} \in [\mathbf{f}], \mathbf{x} \in \mathbf{f}^{-1}(\mathbb{Y}^{\supset}) \wedge \mathbf{x} \notin \mathbf{f}^{-1}(\mathbb{Y}^{\subset}) \\ \Leftrightarrow & \mathbf{x} \notin \bigcap_{\mathbf{f}\in[\mathbf{f}]} \mathbf{f}^{-1}(\mathbb{Y}^{\subset}) \wedge \mathbf{x} \in \bigcup_{\mathbf{f}\in[\mathbf{f}]} \mathbf{f}^{-1}(\mathbb{Y}^{\supset}). \end{array}$$

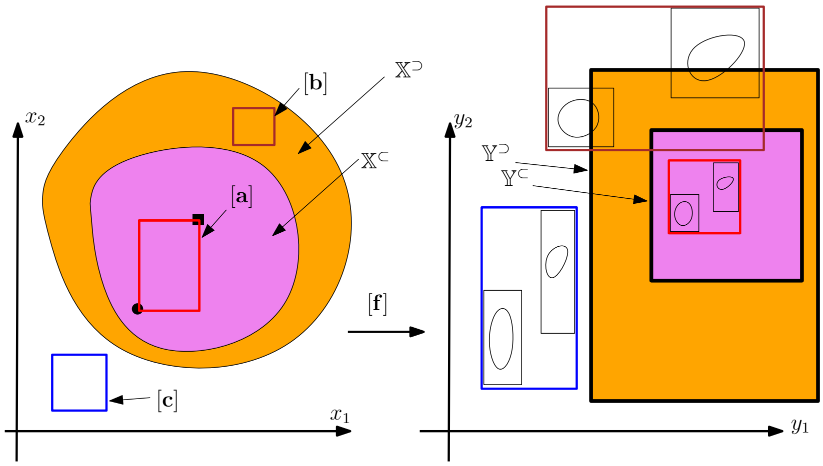

Figure 7: Tests used for the thick set inversion. The box [**a**] is proved to be inside the clear zone $\mathbb{X}^{\subset}$; The box [**b**] is proved to be inside the penumbra $\mathbb{X}^{\supset}\backslash\mathbb{X}^{\subset}$. The box [**c**] is proved to be inside the dark zone, i.e., outside $\mathbb{X}^{\supset}$.

Thus, from (21), we get $[\mathbf{x}] \subset \mathbb{X}^{\supset}\backslash\mathbb{X}^{\subset}$. ■

Convergence

We now provide some convergence properties of our algorithm. We need first to define the convergence of a thick inclusion function $[\![\mathbf{f}]\!]$. This convergence can be interpreted as the continuity of the thick function $[\![\mathbf{f}]\!]([\mathbf{x}])$ around intervals $[\mathbf{x}]$ which are degenerated (i.e., the box $[\mathbf{x}]$ is a singleton).

Definition 13. The thick inclusion function $[\![\mathbf{f}]\!]([\mathbf{x}])$ for $[\mathbf{f}]([\mathbf{x}])$ is said to be *convergent* if for all $\mathbf{a} \in \mathbb{R}^n$, for all sequences of boxes $[\mathbf{x}](k)$ and $[\mathbf{y}](k)$, we have

$$\left.\begin{array}{c} d_H([\mathbf{x}](k),\{\mathbf{a}\}) \overset{k\to\infty}{\longrightarrow} 0 \\ [\mathbf{y}](k) \in [\![\mathbf{f}]\!]([\mathbf{x}](k)) \end{array}\right\} \Rightarrow d_H([\mathbf{y}](k),[\mathbf{f}](\mathbf{a})) \overset{k\to\infty}{\longrightarrow} 0,$$

where d_H is the Hausforff distance between compact sets [35].

Theorem 14. *For a given ε, our algorithm provides three lists $\mathcal{L}^{clear}(\varepsilon)$,$\mathcal{L}^{penumbra}(\varepsilon)$ and $\mathcal{L}^{dark}(\varepsilon)$. Take a point* $\mathbf{a}$. *For ε sufficiently small we have*

$$\begin{array}{llll} (i) & [\mathbf{f}](\mathbf{a}) \subset int\,(\mathbb{Y}^{\subset}) & \Rightarrow & \mathbf{a} \in \bigcup \mathcal{L}^{clear} \\ (ii) & [\mathbf{f}](\mathbf{a}) \subset int\,\left(\overline{\mathbb{Y}^{\supset}}\right) & \Rightarrow & \mathbf{a} \in \bigcup \mathcal{L}^{dark} \\ (iii) & [\mathbf{f}](\mathbf{a}) \cap int\,(\mathbb{Y}^{\supset}\backslash\mathbb{Y}^{\subset}) \neq \emptyset & \Rightarrow & \mathbf{a} \in \bigcup \mathcal{L}^{penumbra} \end{array} \quad (46)$$

where $int(\mathbb{A})$ *denotes the interior of the set* $\mathbb{A}$ *[35].*

PROOF. The proof is by contradiction. Assume that for all ε the box containing $\mathbf{a}$ is never classified. It means that there exists a sequence of boxes $[\mathbf{x}](k)$ converging to $\mathbf{a}$ such that none of the three tests is satisfied for all $[\mathbf{x}](k)$.
(i) Since for all k, $(\llbracket\mathbf{f}\rrbracket([\mathbf{x}](k)) \subset \mathbb{Y}^{\subset})^{\forall}$ is false, there exists a sequence $[\mathbf{y}](k) \in \llbracket\mathbf{f}\rrbracket([\mathbf{x}](k))$ such that $([\mathbf{y}](k) \subset \mathbb{Y}^{\subset})$ is false. Now, since $\llbracket\mathbf{f}\rrbracket$ is a convergent thick inclusion function for $[\mathbf{f}]$, $d_H([\mathbf{y}](k), [\mathbf{f}](\mathbf{a})) \to 0$. Since $int(\mathbb{Y}^{\subset})$ is an open set, we cannot have $[\mathbf{f}](\mathbf{a}) \subset int(\mathbb{Y}^{\subset})$.
(ii) Since for all k, $(\llbracket\mathbf{f}\rrbracket([\mathbf{x}])(k) \cap \mathbb{Y}^{\supset} = \emptyset)^{\forall}$ is false, using the same reasoning as for (i), we get that we cannot have $[\mathbf{f}](\mathbf{a}) \subset int\left(\overline{\mathbb{Y}^{\supset}}\right)$.
(iii) Since for all k,$(\llbracket\mathbf{f}\rrbracket([\mathbf{x}])(k) \cap (\mathbb{Y}^{\supset}\backslash\mathbb{Y}^{\subset}) \neq \emptyset)^{\forall}$ is false, again, we conclude that we cannot have $([\mathbf{f}](\mathbf{a}) \cap int(\mathbb{Y}^{\supset}\backslash\mathbb{Y}^{\subset}) \neq \emptyset)$.
As a consequence, we get that if either (i), (ii) or (iii) is satisfied then $\mathbf{a}$ will be classified inside one of the three lists. Moreover, from Equation (41), we get that $\mathbf{a}$ will be classified on the right list.■

Remark 15. For any box $[\mathbf{y}]$, we always have $[\mathbf{y}] \subset \mathbb{Y}^{\subset}$ or $[\mathbf{y}] \subset \overline{\mathbb{Y}^{\supset}}$ or $[\mathbf{y}] \cap (\mathbb{Y}^{\supset}\backslash\mathbb{Y}^{\subset}) \neq \emptyset$. Moreover, in a generic situation, we have $[\mathbf{y}] \subset int(\mathbb{Y}^{\subset})$ or $[\mathbf{y}] \subset int\left(\overline{\mathbb{Y}^{\supset}}\right)$ or $[\mathbf{y}] \cap int(\mathbb{Y}^{\supset}\backslash\mathbb{Y}^{\subset}) \neq \emptyset$. Therefore, Theorem 14 tells us that the part of the search space which will not be classified are rare. Except in atypical situations, these regions will correspond to the boundaries of the penumbra $\mathbb{X}^{\supset}\backslash\mathbb{X}^{\subset}$.

5. Test-Cases

This section provides five test-cases to illustrate the efficiency of our method. All these test cases solve a thick inversion problem $\llbracket\mathbb{X}\rrbracket = [\mathbf{f}]^{-1}(\llbracket\mathbb{Y}\rrbracket)$. In the figures, all red boxes are shown to be inside the clear zone $\mathbb{X}^{\subset}$; all blue boxes are inside the dark zone, *i.e.*, outside $\mathbb{X}^{\supset}$; all orange boxes are proved to be inside the penumbra. Nothing is known for the small yellow boxes.

Test-case 1. *Thick translation.* Consider the thick set $\llbracket\mathbb{Y}\rrbracket = [\mathbb{Y}^{\subset}, \mathbb{Y}^{\supset}]$ where $\mathbb{Y}^{\subset}, \mathbb{Y}^{\supset}$ are two disks with center $(0,0)$ and with radius $r^{\subset} = 1$ and $r^{\supset} = 2$, respectively. Consider the thick function $[\mathbf{f}](\mathbf{x}) = \mathbf{x} - [\mathbf{v}]$ where $[\mathbf{v}] = [0.7, 1.3] \times [-0.02, 0.02]$. Figure

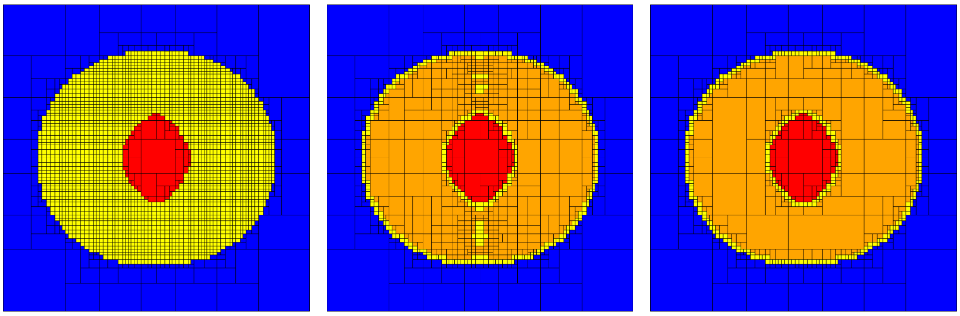

Figure 8: Left: With classical intervals; Center: with subset-supset based thick intervals; Right: with lower-upper bounds based thick intervals

8 represents an approximation of $[\![\mathbb{X}]\!] = [\mathbf{f}]^{-1}([\![\mathbb{Y}]\!])$ using three types of intervals: the classical intervals (left), the thick intervals with a subset-supset representation (center) and the thick intervals defined by lower-upper interval bounds (right). These results have been obtained with $\varepsilon = 0.1$ and the frame box corresponds to $[-2, 4] \times [-2, 4]$. These figures have been generated in 2.1sec for the left figure; 0.21sec for the centered figure and 0.19 for the right figure. As we can see on this figure, the penumbra is better (*i.e.*, without any uncertain boxes inside) characterized with a lower/upper bound representation for the thick intervals.

We compared the computing time (on a processor i5-2520M@2.50GHz) and the number of bisections with the traditional approach (which does not characterize the penumbra) and our method. We get the table below. We observe that when ε is small, classical methods are much less efficient due to the fact many bisections take place inside the penumbra. This observation is also valid for all five test-cases considered in this paper.

	Classical method		Our method	
ε	#bisection	time(s)	#bisection	time(s)
0.5	195	0.0003	159	0.0012
0.1	2215	0.0031	783	0.0047
0.05	8043	0.0130	1623	0.0117
0.01	470407	0.6701	13383	0.1779
0.005	1867707	2.6515	26823	0.5634
0.001	29722487	42.0044	107443	6.1724

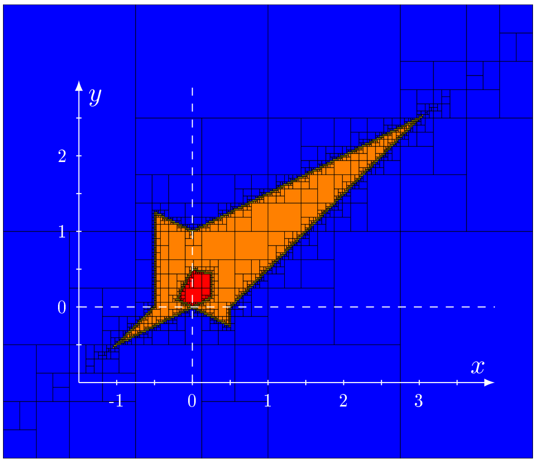

Figure 9: Approximation of the tolerable-united solution sets of the interval linear system of Test-case 2

Test-case 2. *Tolerable-United solution sets.* Consider the interval linear system [21]

$$\begin{pmatrix} [2,4] & [-2,0] \\ [-1,1] & [2,4] \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \begin{pmatrix} [-1,1] \\ [0,2] \end{pmatrix}. \tag{47}$$

The left hand side corresponds to a thick function and the right hand side corresponds to a thin set. The solution set $[\![\mathbb{X}]\!] = [\![\mathbb{X}^{\subset}, \mathbb{X}^{\supset}]\!]$ has for subset bound the *tolerable solution set* $\mathbb{X}^{\subset}$ for supset bound the *united solution set* $\mathbb{X}^{\supset}$ [36] . Some techniques have been developed to approximate these sets [37] in the linear case. They are mainly based on the Kaucher interval arithmetic [38, 39] and may be used to find boxes inside the penumbra. Now, these methods have mainly been developed to deal with linear interval problems and cannot be used to find boxes inside the penumbra for general nonlinear problems as for the two following test-cases. For this example, the thick set-inversion algorithm provides the paving of Figure 9, in less than 0.4 sec, for $\varepsilon = 0.01$.

Test-case 3. *Parameter estimation.* Consider the parametric model

$$y_m(\mathbf{x}, t) = x_1 e^{-x_2 t}, \tag{48}$$

where $\mathbf{x} = (x_1, x_2)$ is the parameter vector and $t \in \mathbb{R}$ is the time. At time t_i, we collect measurements y_i with some interval uncertainties as written in Table 2. Note that one of the main difficulties of this problem is that uncertainties exist on the independent variable (here the time) [40, 41]. In our formulation, the uncertainty of the t_i is stored inside the model under the form of a thick function.

i	$[t_i]$	$[y_i]$
1	$[0.03, 0.06]$	$[4, 8]$
2	$[0.09, 0.12]$	$[2, 6]$
3	$[0.15, 0.18]$	$[2, 5]$
4	$[0.21, 0.24]$	$[1, 3]$
5	$[0.27, 0.3]$	$[0, 2]$

Table 2: Measurements (t_i, y_i) used for estimation

The set of all feasible parameter vectors is

$$\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^2 \mid \forall i \in \{1, \dots, 5\}, \exists t \in [t_i], x_1 e^{-x_2 t} \in [y_i]\}. \tag{49}$$

If we define the thick function

$$[\mathbf{f}](\mathbf{x}) = \begin{pmatrix} x_1 e^{-x_2 [t_1]} \\ \vdots \\ x_1 e^{-x_2 [t_5]} \end{pmatrix} \tag{50}$$

and the box

$$[\mathbf{y}] = [y_1] \times \cdots \times [y_5], \tag{51}$$

then the thick set $[\![\mathbb{X}]\!] = [\![\mathbb{X}^{\subset}, \mathbb{X}^{\supset}]\!] = [\mathbf{f}]^{-1}(\mathbb{Y})$ is composed with the two sets

$$\mathbb{X}^{\subset} = \{\mathbf{x} \in \mathbb{R}^2 \mid \forall i \in \{1, \dots, 5\}, \forall t \in [t_i], x_1 e^{-x_2 t} \in [y_i]\} \tag{52}$$

and

$$\mathbb{X}^{\supset} = \{\mathbf{x} \in \mathbb{R}^2 \mid \forall i \in \{1, \dots, 5\}, \exists t \in [t_i], x_1 e^{-x_2 t} \in [y_i]\}. \tag{53}$$

For $\varepsilon = 0.1$, the thick set-inversion algorithm computes an approximation of the thick set $[\![\mathbb{X}^{\subset}, \mathbb{X}^{\supset}]\!]$ as represented by Figure 10. The left figure is obtained in 1.8 sec and contains 10337 boxes. The right figure is obtained in 0.2 sec and contains 2744 boxes.

Test-case 4. *Communication area.* Consider p marks $\mathbf{m}(i)$ located at position $(m_1(i), m_2(i))$ given by Table 3 and a robot at the position $\mathbf{x} = (x_1, x_2)$.

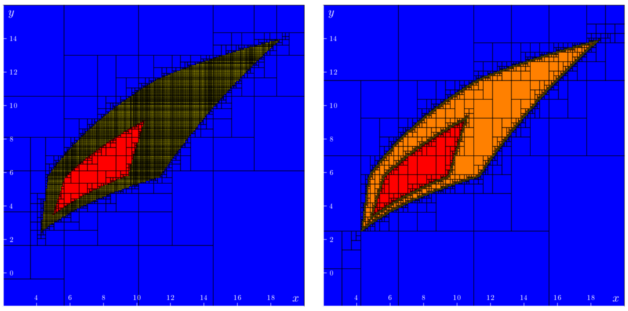

Figure 10: Representation of the thick set $[\![\mathbb{X}^{\subset}, \mathbb{X}^{\supset}]\!]$ associated to the estimation problem. All blue boxes are outside $\mathbb{X}^{\supset}$ and all red boxes are inside $\mathbb{X}^{\subset}$. The orange boxes (on the right figure) are outside $\mathbb{X}^{\subset}$ and inside $\mathbb{X}^{\supset}$. The left Figure, obtained with classical interval tests, do not classify any box in the penumbra. With a thick set inversion approach, we get an inner approximation of the penumbra.

i	1	2	3	4
$\mathbf{m}_1(i)$	1 ± 0.5	10 ± 0.5	10 ± 0.5	-2 ± 0.5
$\mathbf{m}_2(i)$	3 ± 0.5	-1 ± 0.5	6 ± 0.5	-5 ± 0.5

Table 3: Location of the marks

The robot is able to communicate with the mark $\mathbf{m}(i)$ if its distance to the mark is smaller than 10m, *i.e.*, if $\|\mathbf{x} - \mathbf{m}(i)\| < 10$. The communication is not possible if the distance is larger than 20m. With a distance inside $[10, 20]$, the communication is uncertain. The set of all positions for the robot such that the robot can communicate with all marks is a thick set defined by

$$[\![\mathbb{X}]\!] = [\mathbf{f}]^{-1}\left([\![\mathbb{Y}^{\subset}, \mathbb{Y}^{\supset}]\!]\right) \tag{54}$$

where

$$\mathbb{Y}^{\subset} = [0, 10]^{\times 4},\ \mathbb{Y}^{\supset} = [0, 20]^{\times 4} \tag{55}$$

and

$$[f_i](\mathbf{x}) = \|\mathbf{x} - [\mathbf{m}](i)\|. \tag{56}$$

Our thick inversion algorithm provides in less than 0.3 sec, the paving represented on Figure 11. In the example, the clear and the dark boxes could have been obtained using existing interval algorithms. But these methods have to bisect everywhere inside the

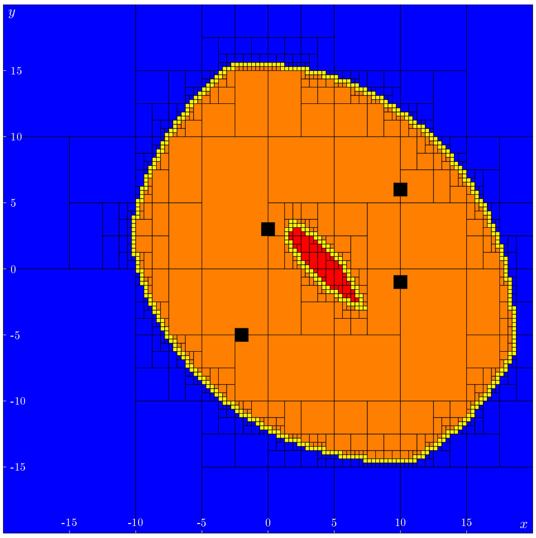

Figure 11: Thick set $[\![\mathbb{X}]\!]$ representing the communication region. The dark zone (blue) corresponds to position where the robot cannot communicate with all marks. In the clear zone (red), the robot is able to communicate with all marks.

penumbra. Using thick interval arithmetic, we are able to conclude for a orange box that there is no need to bisect it.

Test-case 5. *Explored zone.* We now illustrate the thick set inversion on the problem of characterizing a zone explored by a robot [25]. The corresponding experiment has been performed October 20, 2015, on a 46 minutes (=2760 sec) mission in the Road-stead of Brest (France, Brittany) by the underwater mine hunter robot, *Daurade* (see Figure 12), which realized a classical survey pattern composed of a set of parallel tracks. This robot has been built by ECA robotics and used by DGA Tn (Direction Général de l'Armement - Techniques Navales). Daurade is mainly used to secure a zone and check that there is no mine lying on the sea floor. Assessment of the covered area is usually done manually by an operator who looks at the sonar images. The thick inversion algorithm can be used to validate the mission plan or, at the end of the mission, to check the area to be explored has indeed been covered.

For the navigation, *Daurade* relies an Inertial Measurement Unit (Phins II IXBlue) coupled with a DVL (Doppler Velocity Log), which returns after integration every second a box $[\mathbf{a}] \subset \mathbb{R}^2$, which contains $\mathbf{a} = (a_1, a_2)$, the 2D coordinates of the robot expressed in

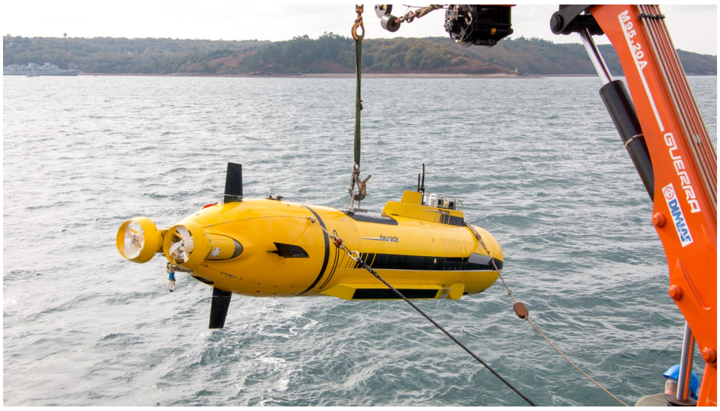

Figure 12: *Daurade*: the underwater robot used for our experiment. Photo: S. Rohou

an absolute frame. The depth is not taken into account since we explore a flat bottom. For this mission, we need to guarantee that the area of interest has been totally explored without any gap. At the beginning of the mission, the position of the robot is exactly known. Once under the water, no GPS data are available and the estimated position of the robot drifts with the time. At the end of the mission, the position accuracy is of 17 meters. At each $t \in \{1, 2, \ldots, m\}$ where $m = 2760$, the visible set (*i.e.*, the part of the bottom which is seen by the sonar of the robot) is a disk of radius 50 meters around the robot, the position of which is not exactly known. The *explored zone* $\mathbb{X}$ corresponds to the union of all patches that have been seen:

$$\mathbb{X} = \bigcup_{t \in [1,2,\ldots,m\}} f_t^{-1}([0, 50[), \tag{57}$$

where

$$f_t(\mathbf{x}) = \sqrt{(x_1 - a_1(t))^2 + (x_2 - a_2(t))^2} \tag{58}$$

The complementary set of $\mathbb{X}$ is

$$\overline{\mathbb{X}} = \bigcap_{t \in \{1,2,\ldots,m\}} f_t^{-1}([50, \infty]) = \mathbf{f}^{-1}([50, \infty]^m),$$

where $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$. The function $\mathbf{f}(\mathbf{x})$ is consistent with the intervals

$[a_1](t)$ and $[a_2](t)$ containing the positions of the robot iff

$$\mathbf{f}(\mathbf{x}) \in [\mathbf{f}](\mathbf{x}) = \begin{pmatrix} \sqrt{(x_1 - [a_1](1))^2 + (x_2 - [a_2](1))^2} \\ \vdots \\ \sqrt{(x_1 - [a_1](m))^2 + (x_2 - [a_2](m))^2} \end{pmatrix}$$

Since the function $[\mathbf{f}](\mathbf{x})$ is thick, the characterization of the thick set $[\![\overline{\mathbb{X}}]\!]$ is a thick set inversion problem which can be characterized by our algorithm. Its complementary $[\![\mathbb{X}]\!]$ can thus be derived. The resulting enclosure of $[\![\mathbb{X}]\!]$, given on Figure 13, is computed in less than 3 minutes. Note that the penumbra (orange) is larger in zones when the estimation of the position of the robot is less accurate. The black tube corresponds to $[\mathbf{a}](t)$. From the information given by the inertial system, for any point $\mathbf{x}_o$ in the orange zone, it is impossible to known if $\mathbf{x}_o$ has been seen or not by the sonar. This ambiguity comes from the uncertainty related to the position $\mathbf{a}(t)$ of the robot.

Remark 16. In [25], the same problem has been considered and a similar approximation has been obtained using a specific algorithm. Whereas our algorithm is able to find an inner approximation of the penumbra, the algorithm in [25] was not able to find such an inner approximation (in the case where the visible zone is a disk, as here). In the paper [25], the problem was not formalized as a thick set inversion problem and the resulting algorithm could not solve other thick set inversion problems such the Test-cases 1,2,3,4 presented in this section.

6. Conclusion

This paper deals with the set-inversion problem $\mathbb{X} = \mathbf{f}^{-1}(\mathbb{Y})$ in the case where both $\mathbf{f}$ and $\mathbb{Y}$ are uncertain, *i.e.*, $\mathbf{f}$ belongs to the interval of functions $[\mathbf{f}^-, \mathbf{f}^+]$ and $\mathbb{Y}$ belongs to a thick set, *i.e.*, an interval of sets $[\![\mathbb{Y}]\!] = [\![\mathbb{Y}^\subset, \mathbb{Y}^\supset]\!]$. After introducing the new notions of thick intervals and thick boxes, a new algorithm for set inversion has been proposed. It is able to compute a thick solution set $[\![\mathbb{X}]\!] = [\![\mathbb{X}^\subset, \mathbb{X}^\supset]\!]$ containing all feasible solution sets.

From the computational point of view, thick intervals allow us to have a better understanding of the uncertainty. For instance, for the set inversion problem, we are able

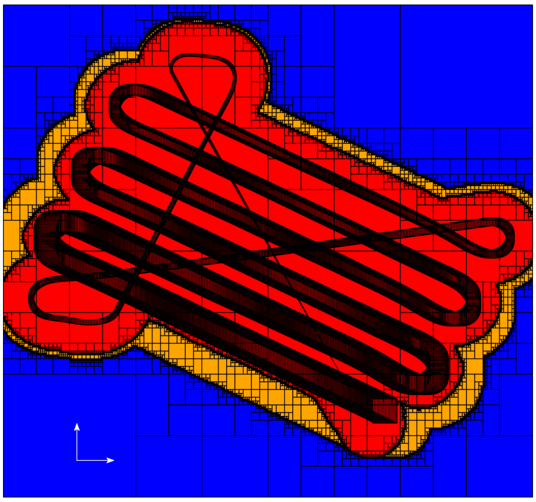

Figure 13: Thick set $[\![\mathbb{X}]\!]$ representing the explored zone. The dark zone (blue) has certainly been unexplored and the clear zone (red) has certainly been explored. Here, certainly means: 'for all feasible trajectories'

to detect that a box is included in the penumbra $\mathbb{X}^{\supset}\backslash\mathbb{X}^{\subset}$. In this penumbra, we can conclude that any bisection would be useless. This could not have not been detected using classical intervals. As a consequence, the accumulation zone (*i.e.*, the part of the search space where tiny boxes are still bisected) for thick interval based algorithms has a zero volume, since it corresponds to the boundary of the penumbra. Using classical intervals instead, we could obtain similar results, but the accumulation zone would correspond to the whole penumbra, which has a nonzero volume. As a result, a large part of the computational burden made by traditional interval algorithms is done on a part of the search space which has no influence on the final result.

Note. The `Python` programs, associated with the test-case are given at the following link.

www.ensta-bretagne.fr/jaulin/thick.html

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