Estimability Characterization Using A New Interval-Based Method

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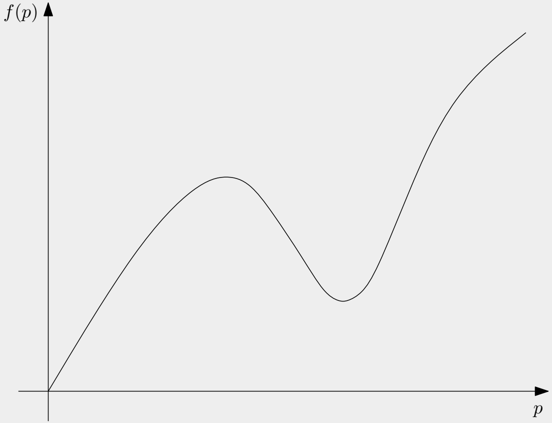
Outline

Estimator Accuracy & Noise Addition

What happens if the model is a nonlinear one ?

Nonlinear model - Identifiability [Braems *et al.*, 2001] - Estimability



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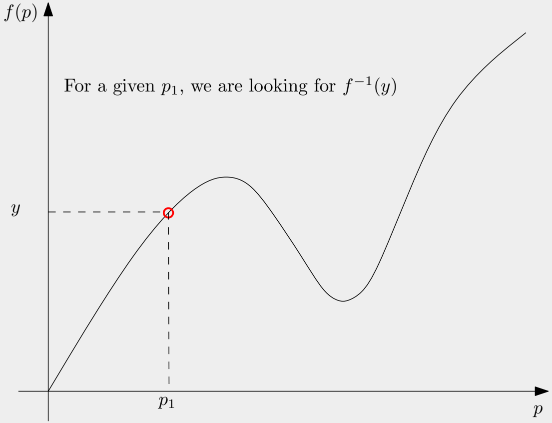

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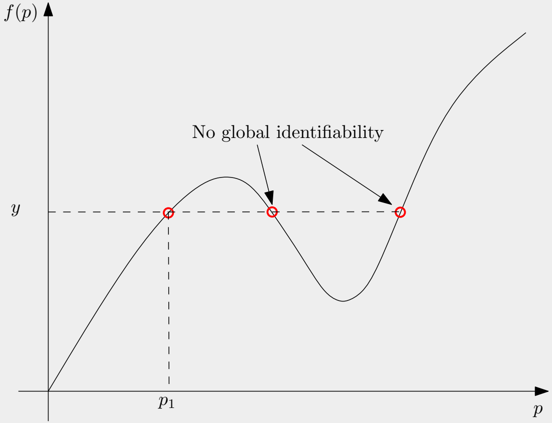

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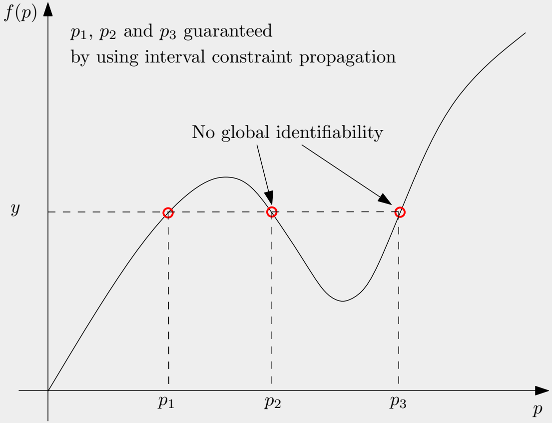

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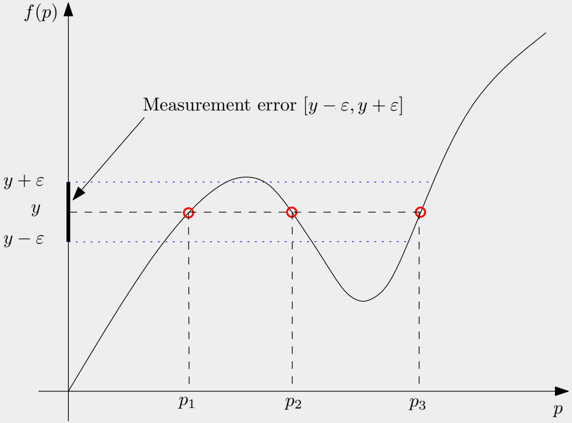

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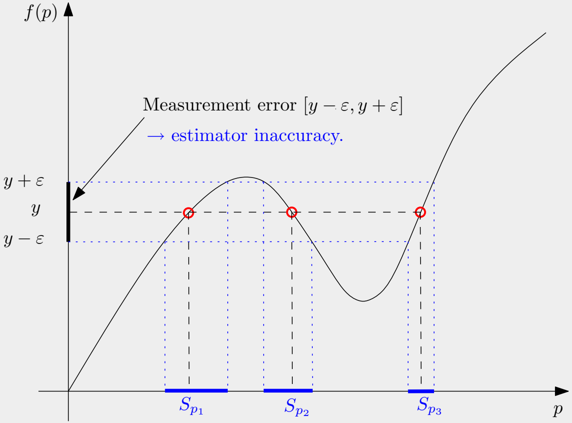

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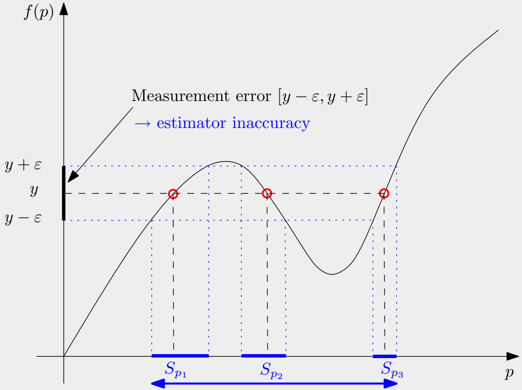

Here, classical methods fail to predict the prior estimator accuracy.

Estimator Accuracy & Noise Addition

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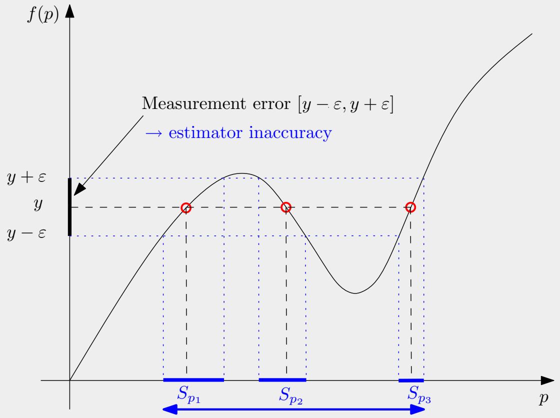


Estimability is defined as the size of the blue set $\{S_{p_1}, S_{p_2}, S_{p_3}\}$.

Estimability Definition

Evaluating The Prior Accuracy of An Estimator

Estimability Concept

- A prior property
- which states on the **accuracy of the parameter estimation**
- in the case of an additive noise.

Why such a concept ?

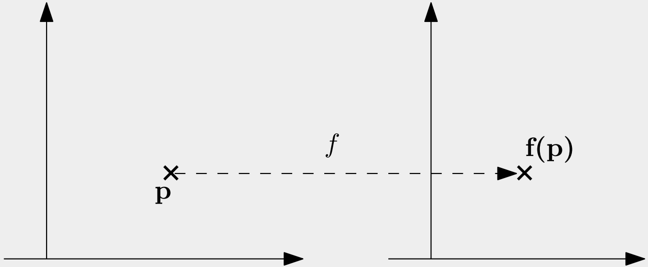
- A parameter vector can be identifiable [Walter & Pronzato, 1997, Lagrange *et al.*, 2007] but poorly estimable for a given model,
- Mostly because experimental data are noisy.

It is important to predict the accuracy of an estimator.

Estimability Function ξ_f

Using Interval analysis, Contraint Propagation and Set Inversion

Direct Image - ε-inflation - Set inversion - Size Operator



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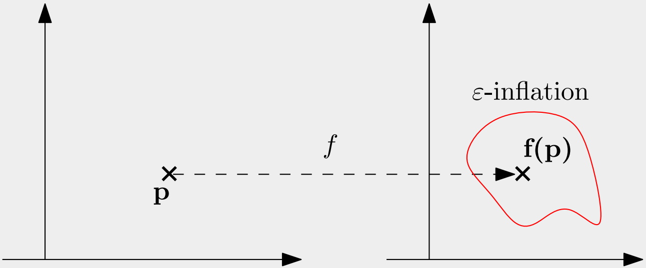

Additive noise $\rightarrow$ ε-inflation

$$\begin{array}{rccc} I_\varepsilon : & \mathbb{R}^n & \rightarrow & \mathbb{IR}^n \\ & \mathbf{p} & \rightarrow & \mathbf{f(p)} + \mathbb{U} \end{array} \quad (1)$$

$\mathbb{U} = [-\varepsilon, \varepsilon]$ is the limited support noise set.

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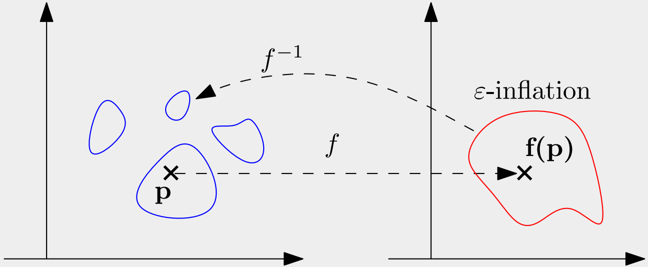

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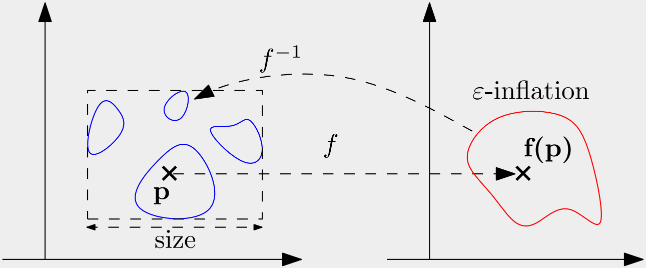


Definition of the estimability function ξ_f

$$\begin{aligned} \xi_f : \quad & \mathbb{R}^n \rightarrow \mathbb{R}^+ \\ & \mathbf{p} \rightarrow \Sigma(\mathbf{f}^{-1}(\mathbf{f}(\mathbf{p}) + \mathbb{U})) \end{aligned} \quad (1)$$

Size operator Σ is positive and monotonic.

Beyond Classiscal Estimation Theory

Assumptions and solutions

Our assumptions are:

- Additive noise has a known limited support $[-\varepsilon, +\varepsilon]$,
- Unknown inverse model,
- Biased and nonlinear estimator
  $\Rightarrow$ Cramer-Rao Lower Bound not suitable

Our solution is to :

1. Compute $f(p)$, the direct image of p,
2. Inflate the result with error $f(p) + [-\varepsilon, +\varepsilon]$,
3. Invert the set $(f(p) + [-\varepsilon, +\varepsilon])$ with contractors,
4. Compute the size of the inverted set.

Estimability : Applications

Design of Experiment with Nonlinear Models

Examples

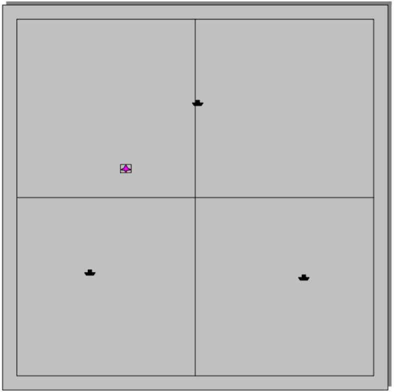
- Physical experiments → in addition to sensitivity analysis,
- Chemical engineering [Berthier *et al.*, 1996, Jayasankar *et al.*, 2009],
- Localization Problems [Reynet *et al.*, 2009].

For design of experiment when :

- Model are nonlinear,
- Reciprocal model are unknown,
- Noise is additive and bounded.

TDOA Passive Location Example

3 receivers - 1 emitter - Time Difference - Hyperbolic equations



Looking for (x, y) from $t_{ij} = t_i - t_j$ measurements.

$$\sqrt{(x - x_i)^2 + (y - y_i)^2} - \sqrt{(x - x_j)^2 + (y - y_j)^2} = ct_{ij}$$

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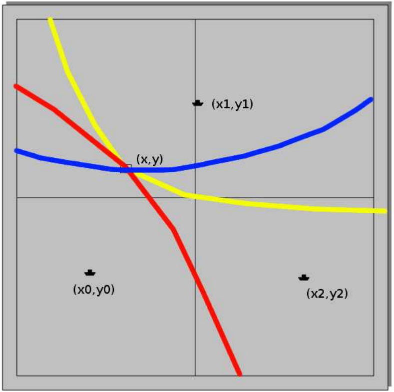

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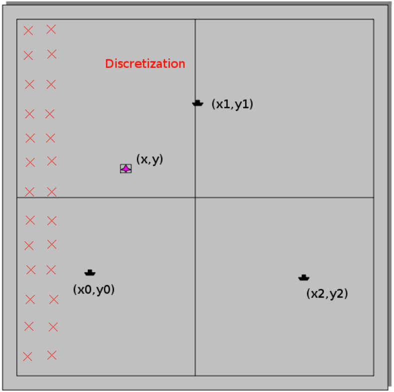


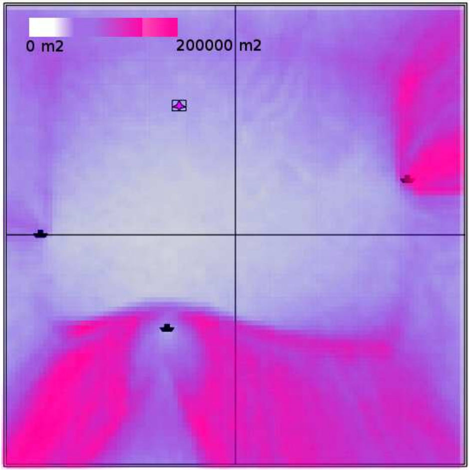
Take a grid to compute estimability ξ_f with :

$$\begin{array}{cccc} f: & \mathbb{R}^2 & \rightarrow & \mathbb{R}^3 \\ & (x, y) & \rightarrow & (t_{01}, t_{12}, t_{20}). \end{array} \quad (2)$$

TDOA Passive Location Estimability

100 x 100 grid example - Time Interval Uncertainty $[-300, 300]$ ns

Search domain : 16 km^2 - Contraction Resolution 50m



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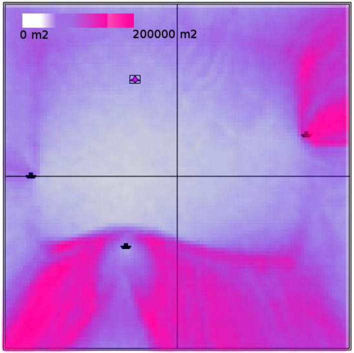


How to read it this estimability grid map?

- Magenta regions correspond to a bad estimation accuracy (i.e. high estimability).
- If looking for a south target, ships should be moved.

A New Goal : bounds on estimability

δ-estimability definition

- For a given parameter $\mathbf{p} \in \mathbb{P}$,
- Find the set $\mathbb{G} = \{\mathbf{q} \in \mathbb{P} \mid \xi_f(\mathbf{p}) - \delta < \xi_f(\mathbf{q}) < \xi_f(\mathbf{p}) + \delta\}$
- with $\delta \in \mathbb{R}^+$
- and ξ_f the estimability function of f defined by :

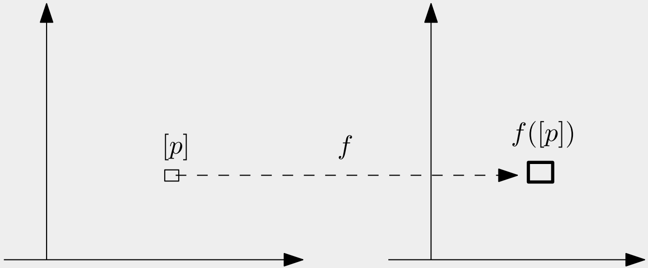
$$\begin{array}{rccc} \xi_f : & \mathbb{R}^n & \rightarrow & \mathbb{R}^+ \\ & \mathbf{p} & \rightarrow & \Sigma(\mathbf{f}^{-1}(\mathbf{f}(\mathbf{p}) + \mathbb{U})) \end{array} \quad (3)$$

Why ?

- To guarantee that the error on estimability does not exceed δ over $\mathbb{G}$.

δ-Estimability Computation Method

Direct image - $[\varepsilon]$-inflation - Contractors



Compute the direct image of $[p]$

δ-Estimability Computation Method

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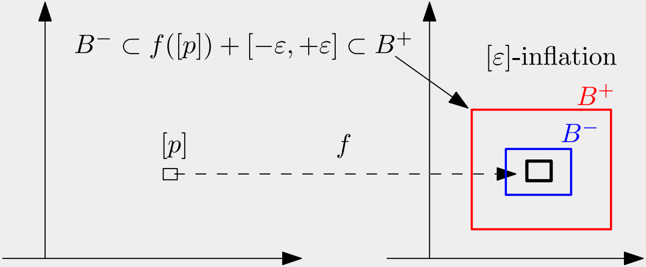

Introduction of an additive noise : **a bounded $[\varepsilon]$-inflation**

$$
\begin{array}{rccc} I_\varepsilon : & \mathbb{R}^n & \rightarrow & \mathbb{R}^n \times \mathbb{R}^n \\ & b & \rightarrow & (B^-, B^+) \end{array} \quad (4)
$$

B^- is outside $f([p])$ if $width(f([p])) < \varepsilon$

δ-Estimability Computation Method

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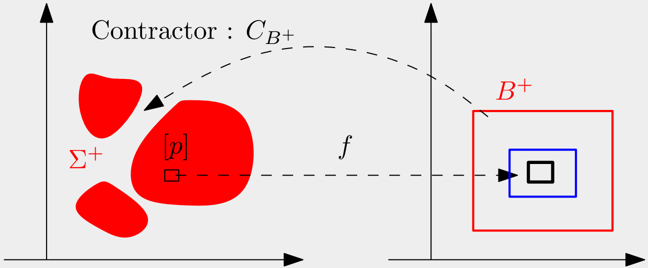

Contractor programming $\rightarrow$ size of the inverted sets

- Size operator $\rightarrow$ estimability bounds Σ^+ and Σ^-
- Adjust $[p]$ size to satisfy the condition $\Sigma^+ - \Sigma^- < \delta$

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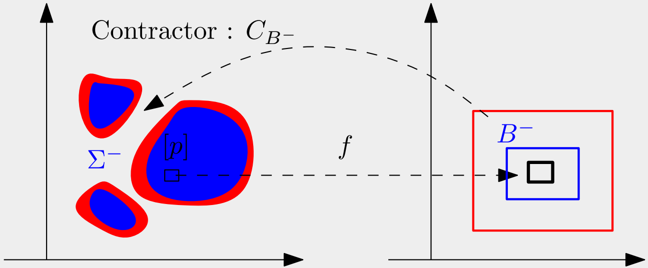

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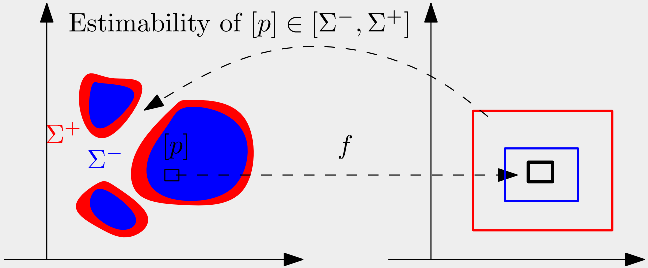

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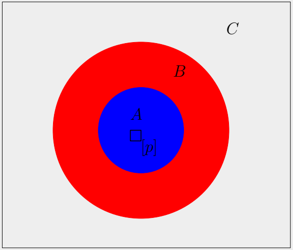
Contractor Programming

How to bound $\Sigma^+ - \Sigma^-$ with contractors [Chabert & Jaulin, 2009] ?

Looking for A, B and C sets



$$A = \mathbb{P} \cap f^{-1}(B^-)$$

$$B = \mathbb{P} \cap f^{-1}(B^+) \setminus A$$

$$C = \mathbb{P} \setminus A \setminus B$$

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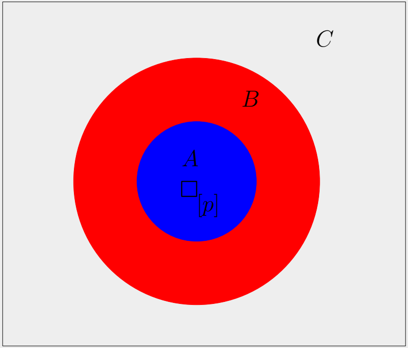
Looking for A, B and C sizes : 4 contractors, 1 computation



- C_{B^+} and $\overline{C_{B^+}}$
- C_{B^-} and $\overline{C_{B^-}}$
- A is given by $\overline{C_{B^-}}$,
- B is given by $\overline{C_{B^+}} \cup C_{B^-}$,
- C is given by C_{B^+}.
- Borders f_{AB} and f_{BC} are not contracted by any contractor, if resolution is reached.

$$\Sigma^+ - \Sigma^- < \Sigma(f_{AB}) + \Sigma(B) + \Sigma(f_{BC})$$

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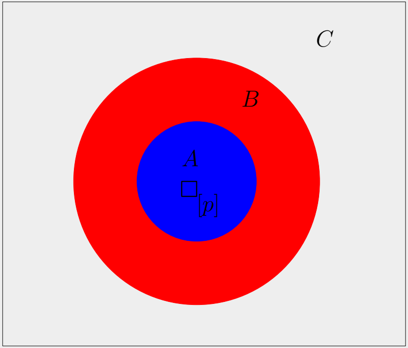

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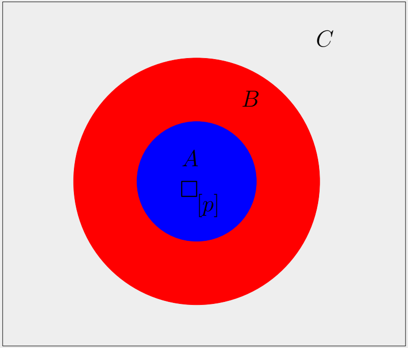

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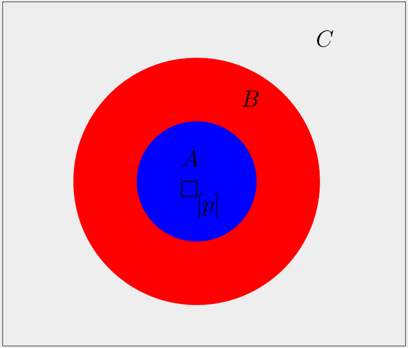

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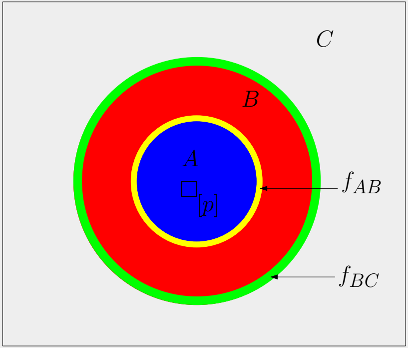

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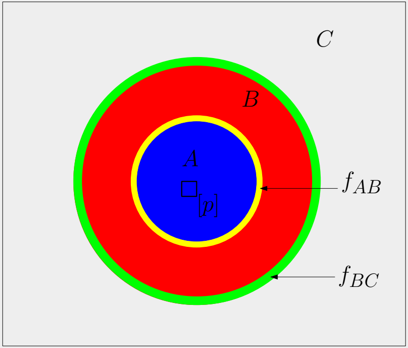

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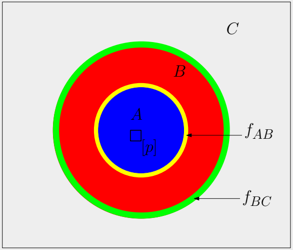
Contractor Programming : Ending

$\Sigma^+ - \Sigma^- < \delta$ and contraction resolution

Refine contraction resolution η and recompute sizes until :

$$\Sigma(f_{AB}) + \Sigma(B) + \Sigma(f_{BC}) < \delta$$



or

$$\eta < width([p])$$

If no condition is satisfied, bissect $[p]$ into $[p_1]$ and $[p_2]$ and start a new computation on these new boxes.

δ-Estimability Algorithm

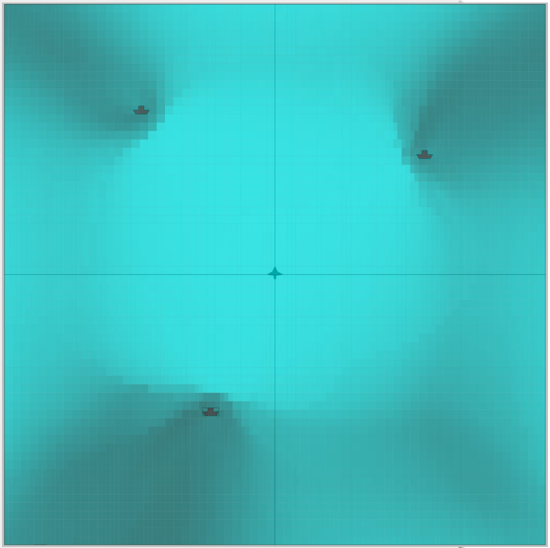
Require: $\varepsilon > 0$ and $\mathbb{P} \subset \mathbb{IR}^n$

Require: $[f]$ monotonic and convergent over $\mathbb{P}$.

push($\mathbb{P}$) into $\mathcal{L}$
while $\mathcal{L} \neq \varnothing$ **do**
    $\mathbf{p} = pop(\mathcal{L})$
    $\mathbf{q} = [f](\mathbf{p})$
    $\mathbf{r} = [\varepsilon]$-inflation($\mathbf{q}$)
    if r is δ-estimable **then**
        push($\mathbf{p}, \Sigma^+, \Sigma^-$) into $\mathcal{L}_{\mathbb{S}}$
    else
        Bisect $\mathbf{p}$ into $\mathbf{p}_1$ and $\mathbf{p}_2$
        push($\mathbf{p}_1$) into $\mathcal{L}$
        push($\mathbf{p}_2$) into $\mathcal{L}$
    end if
end while
Return $\mathcal{L}_{\mathbb{S}}$

Guaranteed Estimability Map

$\varepsilon = 70$ ns - 500 x 500 m^2 - $\delta = 2500$ m^2



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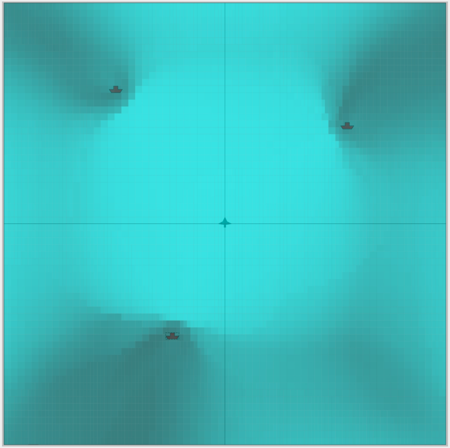

How to read it ?

- The brighter the box, the more accurate the estimation.

Guaranteed Estimability Map

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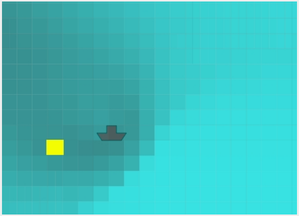

Estimability bounds example : the yellow box

- $[p] = [(-62.5, -54.6875), (-62.5, -54.6875)]$
- Estimability of $[p] \in [925.203, 1715.03]$ m^2

Conclusion

A new horizon for design of experiment

Our method :

- allow to bypass classical estimation theory,
- predict the accuracy of any estimator (even a nonlinear or a biased one),
- faster than gridding,
- gives guarantees on results,
- may be parallelized.

Thank you for your attention

Some questions ?

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