DOA estimation in fluctuating environments: an approximate message-passing approach

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Abstract

In underwater acoustics, wave propagation can be greatly disrupted by random fluctuations in the ocean environment. In particular, phase measurements of the complex pressure field can be heavily noisy and can defeat conventional direction-of-arrival (DOA) estimation algorithms.

In this paper, we propose a new Bayesian approach to address such phase noise through an informative prior on the measurements. This is combined to a sparse assumption on the directions of arrival to achieve a highly-resolved estimation and integrated into a message-propagation algorithm referred to as the "paSAMP" algorithm (for Phase-Aware Swept Approximate Message Passing). Our algorithm can be seen as an extension of the recent phase-retrieval algorithm "prSAMP" to phase-aware priors. xperiments on simulated data mimicking real environments demonstrate that paSAMP outperform conventional approaches (e.g. classic beamforming) in terms of DOA estimation. paSAMP also proves to be more robust to additive noise than other variational methods (e.g. based on mean-field approximation).

Keywords: DOA estimation, sparse representation, Bayesian estimation, variational Bayesian approximations, message passing algorithms

1 Introduction

Common to many applications such as SONAR, RADAR, and telecommunications, direction-of-arrival (DOA) estimation aims at locating one or more sources emitting in some propagation media. Various methods have been proposed to address this problem. They can be distinguished by the assumptions made on the propagating medium and sources.

The beamforming approach [1] constitutes the most famous approach. As it implicitly assumes the noise to be Gaussian and additive, it leads to poor estimation performance for compelex phase perturbations.The so-called "high-resolution" techniques consider additional assumptions over the number or the nature of the sources. This is the case of the well-known MUSIC method [2]. MUSIC assumes the number of sources to be known and the separability of the sub-spaces where the noise and the signal live. More recently, "compressive" beamforming approaches proposed *e.g.* in [3] benefit from an explicit sparse model on the sources.

While all the previously cited approaches rely on an additive Gaussian noise model, recent work has focused on the integration of phase-noise models better accounting for complex propagation processes. Such approaches aim to take into account the wave-front distortion occurring when waves travel through fluctuating media. This is of key interest for a wide range of application fields including as underwater acoustics [4, 5] or atmospheric sound propagation [6, 7]. These contributions mainly relate to recent advances in phase recovery (see e.g. [8, 9, 10, 11]) and the use of informative priors on the missing phases. In this respect, we can mention the Bayesian approach "paVBEM" based on a mean-field approximation [12].

Here, we further explore a variational Bayesian approach. Knowing that higher-order approximations and associated message-passing algorithms outperform mean-field approximations for a wide range of inverse problems [13], we propose a novel approach based on the "swept approximate message passing" (SwAMP) algorithm introduced in [14]. Our algorithm is proven to be more robust to additive noise and multiplicative phase noise than previous approaches using phase-aware priors such as the paVBEM approach [12] and those using non-informative phase priors [9].

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2 PROBLEM STATEMENT

In this section, we recall the Bayesian modeling introduced in [12], which we shall follow throughout of this paper, and introduce the estimation problem we propose to solve.

2.1 Observation Model

Our objective is to design an algorithm able to recover the directions of arrival of a few waves, despite a structured phase-noisy environment, exploiting one single temporal snapshot on a uniform linear sensor array. In underwater acoustics, this noise is mainly due to internal waves, changing the local sound-speed (see *e.g.* [4]). These internal waves and their impact on the acoustic measurements have been studied in different works (see [4, 5]), which leads to a statistical characterization of the phase noise.

In this context, we propose the following observation model: we consider a linear antenna composed of N regularly-spaced sensors and assume that the received signal at sensor n can be expressed as

$$y_n = e^{j\theta_n} \sum_{m=1}^{M} d_{nm} x_m + \omega_n, \tag{1}$$

where θ_n stands for the phase noise due to the propagation through the fluctuating medium and ω_n an additive noise. The scalar d_{nm} is the n-th element of the steering vector $\mathbf{d}_m = [e^{j\frac{2\pi}{\lambda}\Delta\sin(\phi_m)} \dots e^{j\frac{2\pi}{\lambda}\Delta N\sin(\phi_m)}]^T$ where the ϕ_m's are some potential angles of arrival, Δ is the distance between two adjacent sensors and λ is the wavelength of the propagation waves.

Within model (1), at each sensor of the antenna, we assume that the received field is a combination of a few waves arriving from different angles ϕ_m. The DOA estimation problem then consists in identifying the positions of the non-zero coefficients in $\mathbf{x} \triangleq [x_1 \dots x_M]^T$. In underwater acoustics, the phase noise considered in (1) is well-suited to characterize phase perturbations of the wave front in a fluctuating ocean [5], especially in the case of the so-called "partially saturated " propagation regime defined in [4]. This regime focuses on far-field propagation at high frequency with micro-multipaths only. In this case, amplitude variations of the measured acoustic field can be neglected regarding the high sensibility to a structured phase-noise. Note that a similar fluctuation regime has been also identified in atmospheric sound propagation (see [7]).

2.2 Bayesian formulation of the problem

We address the estimation of $\mathbf{x}$ from the measurements $\mathbf{y} \triangleq [y_1, \dots, y_N]^T$ in the presence of (unknown) additive noise $\boldsymbol{\omega} \triangleq [\omega_1, \dots, \omega_N]^T$ and multiplicative phase noise $\boldsymbol{\theta} \triangleq [\theta_1, \dots, \theta_N]^T$. To solve this problem, we consider a Bayesian framework and define some prior assumptions on the different variables in (1).

A first assumption is set on the number of sources (*i.e.* the non-zero coefficients in $\mathbf{x}$) that we suppose to be small in front of the number of sensors. To take into account this sparse hypothesis, we adopt a Bernoulli-Gaussian model $\forall m \in \{1, \dots, M\}$

$$p(x_m) = \rho\, \mathcal{CN}(x_m; m_x, \sigma_x^2) + (1-\rho)\delta_0(x_m), \tag{2}$$

where ρ is the Bernoulli parameter and equals the probability for x_m to be non-zero[1],

Previous studies of the statistical characterization of fluctuation phenomena [4, 5] provide the basis for the definition of a phase-noise prior. In underwater acoustics, [4, 5] exhibited and characterized the existence of a spatial correlation of the measured field all along the antenna. To account for the resulting coherence length, we consider a Markovian model as:

$$p(\theta_n|\theta_{n-1}) = \mathcal{N}(\theta_n; \beta\,\theta_{n-1}, \sigma_\theta^2), \quad \forall n \in \{2, \dots, N\}, \tag{3}$$

$$p(\theta_1) = \mathcal{N}(\theta_1; 0, \sigma_1^2), \tag{4}$$

with $\beta \in \mathbb{R}_+$. Variance σ_θ^2 is related to the coherence length and accounts for the strength of the fluctuations. As an example, a large σ_θ^2 models strong fluctuations of the medium and results in a small coherence length, such that the phase noise varies widely from a sensor to the neighboring ones.

We also introduce an additive noise $\boldsymbol{\omega}$ to account for the combination of a large number of random parasitic phenomena. Based on the central limit theorem, we consider with a classic zero-mean Gaussian distribution with variance σ^2.

Within model (1)-(4), we focus on the following Minimum Mean Square Error (MMSE) problem:

$$\hat{\mathbf{x}} = \underset{\tilde{\mathbf{x}}}{\operatorname{argmin}} \int_x ||\mathbf{x} - \tilde{\mathbf{x}}||_2^2 \; p(\mathbf{x}|\mathbf{y}) d\mathbf{x} \tag{5}$$

[1] We assume the Bernoulli parameter to be the same for each $m \in \{1, \dots, M\}$.

where $\mathbf{y} \triangleq [y_1 \dots y_N]^T$ and $p(\mathbf{x}|\mathbf{y}) = \int_{\boldsymbol{\theta}} p(\mathbf{x}, \boldsymbol{\theta}|\mathbf{y}) d\boldsymbol{\theta}$, with $\boldsymbol{\theta} \triangleq [\theta_1 \dots \theta_N]^T$.

To solve efficiently this problem, we propose to exploit a variational Bayesian inference strategy, that approximates the posterior joint distribution $p(\mathbf{x}, \boldsymbol{\theta}|\mathbf{y})$ by a distribution having a suitable factorization. In [12], a mean-field approximation $p(\mathbf{x}, \boldsymbol{\theta}|\mathbf{y}) \simeq q(\boldsymbol{\theta}) \prod_{m=1}^{M} q(x_m)$ was considered. Here, we address a different type of factorization, called the Bethe approximation, relating to the “approximate message passing” (AMP) algorithms [13]. This approximation exploits higher-order terms which result in better estimation performance [13].

We motivate and detail our approach in the next section.

3 The “paSAMP” algorithm

In this section, we motivate and present the novel algorithm proposed to solve problem (5).

3.1 Motivation and main principles of the approach

AMP algorithms have been considered for a few years as a serious solution to linear problems under sparsity constraints. First considered in the sole case of *i.i.d* (sub-)Gaussian matrices, they have been recently extended to random but more general matrices by the “vector approximate message passing” (VAMP) algorithm [15] and to highly correlated matrices by the “swept approximate message passing” (SwAMP) approach [14]. Both methods aim at alleviating the convergence issues of AMP (notably highlighted in [16]) due to its parallel update structure.

AMP, VAMP and SwAMP have been extended to generalized but component-wise measurement models [17, 18, 14]. They have been then successfully applied to the phase recovery task where $\theta_n \sim \mathcal{U}[0, 2\pi]$, $\forall n \in \{1, \dots, N\}$, giving raise to the so-called “prGAMP” [19], “prVAMP” [10] and “prSAMP” [9] algorithms. In particular, the latter was shown to outperform other state-of-the-art algorithms among which the mean-field approximation [8].

The prSAMP algorithm constitutes thus a promising approach for our DOA estimation[2] problem (5). However, here, the phases θ_n's are spatially-correlated (as represented in the Markov model). This prevents us from a direct application of prSAMP.

We thus propose an iterative algorithm based on the two following mathematical derivations:

i) the extension of prSAMP to a *i.i.d.* Gaussian prior on the phases,

ii) the use of a mean-field approximation to estimate the (Gaussian) posterior distribution on the phases.

We detail both aspects in the next two sub-sections. In the following, we refer to the proposed procedure as “paSAMP” for “phase-aware SwAMP algorithm”. The pseudo-code of paSAMP is presented in **Algorithm 1**.

3.2 Extension of prSAMP to i.i.d. Gaussian phases

For a sake of clarity and due to space limitation, we will adopt and refer the reader to the notations of paper [14] which introduced the SwAMP algorithm described in **Algorithm 1**.

We would like to adapt this algorithm in order to fit our noise model and the prior on our signal $\mathbf{x}$.

The work of [14] evoke the possibility to obtain a generalized version of SwAMP by replacing $\frac{y_n - \mu_{z_n}(t)}{\sigma^2 + \Sigma_{z_n}(t)}$ in g_n and the $\frac{1}{\Sigma_{z_n}(t)+\sigma^2}$ term in the $\mu_m^r(t)$ by $g_{out}(\omega, V)$ and $-g'_{out}(\omega, V)$, where $g_{out}(\omega, V)$ and $g'_{out}(\omega, V)$ are respectively the moment of order 1 and 2 of the following pdf, assuming that the z_n's follow Gaussian distributions of mean μ_{z_n} and variance Σ_{z_n} as mixtures of Bernoulli-Gaussian distributions.

$$p(z_n|y_n, \mu_{z_n}, \Sigma_{z_n}) = \frac{p(y_n|z_n)\mathcal{CN}(z_n; \mu_{z_n}, \Sigma_{z_n})}{\int_{z'_n} p(y_n|z'_n)\mathcal{CN}(z'_n; \mu_{z_n}, \Sigma_{z_n})} = \frac{p(y_n|z_n)\mathcal{CN}(z_n; \mu_{z_n}, \Sigma_{z_n})}{Z_{nor}} \tag{6}$$

[2] Note in addition that the DOA estimation problem involves a highly-correlated matrix. This further motivates a SwAMP-like approach.

Algorithm 1 prSAMP Algorithm

Input: $\mathbf{y}$, $\mathbf{D}$, σ^2, σ_x^2,T_{max}
Define:
$g_{out,n} \triangleq \frac{y_n - \mu_{z_n}^{m+1}(t)}{\sigma^2 + \Sigma_{z_n}^{m+1}}$
$g'_{out,n} \triangleq \frac{-1}{\Sigma_{z_n} + \sigma^2}$
$g_{in,m} \triangleq E_{X|Y}\{x_m|\mu_{x_m}, \Sigma_{x_m}\}$
$g'_{in,m} \triangleq var_{X|Y}\{x_m|\mu_{x_m}, \Sigma_{x_m}\}$
while $t < T_{max}$ **do**
  for $n = 1 \dots N$ **do**
    $\hat{z}_n(t) = \sum_{m=1}^{M} d_{nm} a_m(t)$
    $\Sigma_{z_n}^1(t+1) = \sum_{m=1}^{M} |d_{nm}|^2 v_m(t)$
    $\mu_{z_n}^1(t+1) = \hat{z}_n(t) - \Sigma_{z_n}^1(t) g_{out,n}$
  end for
  for $m = permute[1 \dots M]$ **do**
    $\Sigma_{x_m}(t+1) = (-\sum_{n=1}^{N} |d_{nm}|^2 g'_{out,n})^{-1}$
    $\mu_{x_m}(t+1) = a_m(t) + \Sigma_{x_m}(t+1) \sum_{n=1}^{N} d_{nm} g_{out,n}$
    $v_m(t+1) = \Sigma_{x_m}(t+1) g'_{in,m}$
    $a_m(t+1) = g_{in,m}$
    for $n = 1 \dots N$ **do**
      $\Sigma_{z_n}^{m+1}(t+1) =$
        $\Sigma_{z_n}^{m}(t+1) + |d_{nm}|^2 (v_m(t+1) - v_m(t))$
      $\mu_{z_n}^{m+1}(t+1) =$
        $\mu_{z_n}^{m}(t+1) + d_{nm}(a_m(t+1) - a_m(t))$
        $-g_{out,n}(t)(\Sigma_{z_n}^{m+1}(t+1) - \Sigma_{z_n}^{m}(t+1))$
    end for
    update σ^2 according to [8].
    update $[\mu_{\theta_n}, \Sigma_{\theta_n}]$ according to (44,45).
  end for
end while
Output: $\{\hat{x}_m = a_m(T_{max})\}_m$

First by computing Z_{nor}:

$$
\begin{aligned}
Z_{nor} &= \int_{z'_n} p(y_n|z'_n)\mathcal{CN}(z'_n; \mu_{z_n}, \Sigma_{z_n}) && (7)\\
&= \int_{z'_n,\theta_n} p(y_n|z'_n, \theta_n)p(\theta_n)\mathcal{CN}(z'_n; \mu_{z_n}, \Sigma_{z_n}) && (8)\\
&= \int_{z'_n,\theta_n} p(\theta_n)\mathcal{CN}(z'_n; ye^{-j\theta_n}, \sigma^2)\mathcal{CN}(z'_n; \mu_{z_n}, \Sigma_{z_n}) && (9)\\
&= \int_{z'_n,\theta_n} p(\theta_n)\mathcal{CN}(y_n e^{-j\theta_n}; \mu_{z_n}, \Sigma_{z_n} + \sigma^2)\mathcal{CN}(z'_n; \frac{y_n e^{-j\theta_n}\Sigma_{z_n} + \mu_{z_n}\sigma^2}{\sigma^2 + \Sigma_{z_n}}, \frac{1}{\frac{1}{\sigma^2} + \frac{1}{\Sigma_{z_n}}}) && (10)\\
&= \int_{\theta_n} p(\theta)\mathcal{CN}(y_n e^{-j\theta_n}; \mu_{z_n}, \Sigma_{z_n} + \sigma^2) \int_{z'_n} \mathcal{CN}(z'_n; \frac{y_n e^{-j\theta_n}\Sigma_{z_n} + \mu_{z_n}\sigma^2}{\sigma^2 + \Sigma_{z_n}}, \frac{1}{\frac{1}{\sigma^2} + \frac{1}{\Sigma_{z_n}}}) && (11)\\
&= \int_{\theta_n} p(\theta_n)\mathcal{CN}(y_n e^{-j\theta_n}; \mu_{z_n}, \Sigma_{z_n} + \sigma^2) && (12)\\
&= \int_{\theta_n} \frac{1}{\sqrt{2\pi(\Sigma_{z_n} + \sigma^2)}} \exp\left[-\frac{|y_n e^{-j\theta_n} - \mu_{z_n}|^2}{2(\Sigma_{z_n} + \sigma^2)}\right] p(\theta_n) && (13)\\
&= \exp\left[-\frac{|y_n|^2 + |\mu_{z_n}|^2}{2(\Sigma_{z_n} + \sigma^2)}\right] \frac{1}{\sqrt{2\pi(\Sigma_{z_n} + \sigma^2)}} \int_{\theta_n} \exp\left[\frac{|y_n||\mu_{z_n}| cos(\arg(y_n^* \mu_{z_n}) + \theta_n)}{\Sigma_{z_n} + \sigma^2}\right] p(\theta_n) && (14)
\end{aligned}
$$

Using Von Mises approximations [20] we approximate the cosinus part considering small a:

$$
\frac{1}{\sqrt{2\pi a}} e^{-\frac{1}{2a}(x - x_m)^2} \approx \frac{1}{\pi I_0(\frac{1}{a})} e^{\frac{1}{a} cos(x - x_m)} \qquad (15)
$$

$$Z_{nor} = \exp\Big[-\frac{|y_n|^2+|\mu_{z_n}|^2}{2(\Sigma_{z_n}+\sigma^2)}\Big]\frac{\pi}{\sqrt{2\pi(\Sigma_{z_n}+\sigma^2)}}I_0\Big(\frac{|y_n||\mu_{z_n}|}{\Sigma_{z_n}+\sigma^2}\Big) \tag{16}$$

$$\int_{\theta_n}\mathcal{N}\Big(\theta_n;-\arg(y_n^*\mu_{z_n}),\frac{V+\sigma^2}{|y_n||\mu_{z_n}|}\Big)\mathcal{N}(\theta_n,\mu_{\theta_n},\Sigma_{\theta_n}). \tag{17}$$

$$= \exp\Big[-\frac{|y_n|^2+|\mu_{z_n}|^2}{2(\Sigma_{z_n}+\sigma^2)}\Big]\frac{\pi}{\sqrt{2\pi(\Sigma_{z_n}+\sigma^2)}}I_0\Big(\frac{|y_n||\mu_{z_n}|}{\Sigma_{z_n}+\sigma^2}\Big) \tag{18}$$

$$\int_{\theta_n}\mathcal{N}\Big(\theta_n;\mu^z_{\theta_n},\Sigma^z_{\theta_n}\Big)\mathcal{N}(-\arg(y_n^*\mu_{z_n});\theta_n,\alpha+\sigma_{\theta_n}) \tag{19}$$

with,

$$\frac{1}{\Sigma^z_\theta}=\frac{1}{\alpha}+\frac{1}{\Sigma_{\theta_n}},\quad \mu^z_{\theta_n}=\frac{-\frac{\arg(y_n^*\mu_{z_n})}{\alpha}+\frac{\mu_{\theta_n}}{\Sigma_{\theta_n}}}{\frac{1}{\alpha}+\frac{1}{\Sigma_{\theta_n}}},\quad \alpha=\frac{\Sigma_{z_n}+\sigma^2}{|y_n||\mu_{z_n}|},$$

Finally:

$$Z_{nor} = \exp\Big[-\frac{|y_n|^2+|\mu_{z_n}|^2}{2(\Sigma_{z_n}+\sigma^2)}\Big]\frac{\pi}{\sqrt{2\pi(\Sigma_{z_n}+\sigma^2)}}I_0\Big(\frac{|y_n||\mu_{z_n}|}{\Sigma_{z_n}+\sigma^2}\Big) \tag{20}$$

$$\frac{1}{\sqrt{2\pi(a+\sigma^2_{\theta_n})}}\exp\Big[-\frac{|\mu_{\theta_m}+\arg(y_n^*\mu_{z_n})|^2}{2(a+\sigma^2_{\theta_n})}\Big] \tag{21}$$

Now we can compute the momentum by integrating over the realizations of z_m and over θ_m:

$$E_{Z|Y}\{z_n|y_n,\mu_{z_n},\Sigma_{z_n}\} = \frac{1}{Z_{nor}}\int_{\theta_n}\int_{z_n} z_n\mathcal{N}(z_n;y_ne^{-j\theta_n},\sigma^2)\mathcal{N}(z_n,\mu_{z_n},\Sigma_{z_n}) \tag{22}$$

$$= \frac{1}{Z_{nor}}\int_{\theta_n}\mathcal{N}(y_ne^{-j\theta_n};\mu_{z_n},\Sigma_{z_n}+\sigma^2)p(\theta_n) \tag{23}$$

$$\int_{z_n} z_n\mathcal{N}\Big(z_n;\Big[\frac{y_ne^{-j\theta_n}\Sigma_{z_n}+\mu_{z_n}\sigma^2}{\sigma^2+\Sigma_{z_n}}\Big],\frac{1}{\frac{1}{\sigma^2}+\frac{1}{\Sigma_{z_n}}}\Big) \tag{24}$$

$$= \frac{1}{Z_{nor}}\int_{\theta_n}\mathcal{N}(y_ne^{-j\theta_n};\mu_{z_n},\Sigma_{z_n}+\sigma^2)p(\theta_n)\Big[\frac{y_ne^{-j\theta_n}\Sigma_{z_n}+\mu_{z_n}\sigma^2}{\sigma^2+\Sigma_{z_n}}\Big] \tag{25}$$

$$= \frac{1}{Z_{nor}}\Big[\frac{y_n\Sigma_{z_n}}{\sigma^2+\Sigma_{z_n}}\Big]\int_{\theta_n}e^{-j\theta_n}\mathcal{N}(y_ne^{-j\theta_n};\mu_{z_n},\Sigma_{z_n}+\sigma^2)p(\theta_n)+\Big[\frac{\mu_{z_n}\sigma^2}{\sigma^2+\Sigma_{z_n}}\Big] \tag{26}$$

$$= \frac{1}{Z_{nor}}\Big[\frac{y_n\Sigma_{z_n}}{\sigma^2+\Sigma_{z_n}}\Big]\exp\Big[-\frac{|y_n|^2+|\mu_{z_n}|^2}{2(\Sigma_{z_n}+\sigma^2)}\Big]\frac{1}{\sqrt{2\pi(\Sigma_{z_n}+\sigma^2)}} \tag{27}$$

$$\int_{\theta_n}\exp(-j\theta)\exp\Big[\frac{|y_n||\mu_{z_n}|\cos(\arg(y_n^*\mu_{z_n})+\theta_n)}{\Sigma_{z_n}+\sigma^2}\Big]p(\theta_n)+\Big[\frac{\mu_{z_n}\sigma^2}{\sigma^2+\Sigma_{z_n}}\Big] \tag{28}$$

$$\tag{29}$$

Again by identifying with a Von Mises Distribution :

$$E_{Z|Y}\{z_n|y_n,\mu_{z_n},\Sigma_{z_n}\} = \frac{1}{Z_{nor}}\Big[\frac{y_n\Sigma_{z_n}}{\sigma^2+\Sigma_{z_n}}\Big]\exp\Big[-\frac{|y_n|^2+|\mu_{z_n}|^2}{2(\Sigma_{z_n}+\sigma^2)}\Big]\frac{\pi}{\sqrt{2\pi(\Sigma_{z_n}+\sigma^2)}}I_0\Big(\frac{|y_n||\mu_{z_n}|}{\Sigma_{z_n}+\sigma^2}\Big) \tag{30}$$

$$\int_{\theta_n}e^{-j\theta_n}\mathcal{N}\Big(\theta_n;\mu^z_{\theta_n},\Sigma^z_{\theta_n}\Big)\mathcal{N}(-\arg(y_n^*\mu_{z_n});\theta_n,a+\sigma_{\theta_n})+\Big[\frac{\mu_{z_n}\sigma^2}{\sigma^2+\Sigma_{z_n}}\Big] \tag{31}$$

by variable change $\theta_n+\mu^z_{\theta_n}\leftarrow\theta_n$

$$E_{Z|Y}\{z_n|y_n,\mu_{z_n},\Sigma_{z_n}\} = \frac{1}{Z_{nor}}\Big[\frac{y_n\Sigma_{z_n}}{\sigma^2+\Sigma_{z_n}}\Big]\exp\Big[-\frac{|y_n|^2+|\mu_{z_n}|^2}{2(\Sigma_{z_n}+\sigma^2)}\Big]\frac{\pi}{\sqrt{2\pi(\Sigma_{z_n}+\sigma^2)}}I_0\Big(\frac{|y_n||\mu_{z_n}|}{\Sigma_{z_n}+\sigma^2}\Big) \tag{32}$$

$$\frac{1}{\sqrt{2\pi(a+\sigma^2_{\theta_n})}}\exp\Big[-\frac{|\mu_{\theta_m}-\arg(y_n^*\mu_{z_n})|^2}{2(a+\sigma^2_{\theta_n})}\Big]\int_{\theta_n}\exp(-j(\theta_n+\mu^z_{\theta_n})) \tag{33}$$

$$\frac{1}{\sqrt{2\pi\Sigma^z_{\theta_n}}}\exp\Big(-\frac{\theta_n^2}{2\Sigma^z_{\theta_n}}\Big)+\Big[\frac{\mu_{z_n}\sigma^2}{\sigma^2+\Sigma_{z_n}}\Big] \tag{34}$$

thanks to Von Mises identification :

$$E_{Z|Y}\{z_n|y_n,\mu_{z_n},\Sigma_{z_n}\} = \frac{1}{Z_{nor}}\left[\frac{y_n\Sigma_{z_n}}{\sigma^2+\Sigma_{z_n}}\right]\exp\left[-\frac{|y_n|^2+|\mu_{z_n}|^2}{2(\Sigma_{z_n}+\sigma^2)}\right]\frac{\pi}{\sqrt{2\pi(\Sigma_{z_n}+\sigma^2)}}I_0\left(\frac{|y_n||\mu_{z_n}|}{\Sigma_{z_n}+\sigma^2}\right) \tag{35}$$
$$\frac{\exp\left(-j\mu^z_{\theta_n}\right)}{\sqrt{2\pi(a+\sigma^2_{\theta_n})}}\exp\left[-\frac{|\mu_{\theta_m}-\arg(y_n^*\mu_{z_n})|^2}{2(a+\sigma^2_{\theta_n})}\right]\mathbf{R_0}\left(\frac{1}{\Sigma^z_\theta}\right)+\left[\frac{\mu_{z_n}\sigma^2}{\sigma^2+\Sigma_{z_n}}\right]$$

After simplification with Z_{nor} we obtain:

$$E_{Z|Y}\{z_n|y_n,\mu_{z_n},\Sigma_{z_n}\} = \frac{y_n\Sigma_{z_n}e^{-j\mu^z_{\theta_n}}}{\sigma_n^2+\Sigma_{z_n}}\mathbf{R_0}\left(\frac{1}{\Sigma^z_\theta}\right)+\frac{\mu_{z_n}\sigma^2}{\sigma^2+\Sigma_{z_n}}, \tag{36}$$

By similar method we obtain :

$$var_{Z|Y}\{z_n|y_n,\mu_{z_n},\Sigma_{z_n}\} = \frac{|y_n\Sigma_{z_n}|^2+|\mu_{z_n}\sigma^2|^2}{|\sigma^2+\Sigma_{z_n}|^2}+\frac{|y_n\Sigma_{z_n}||\mu_{z_n}\sigma^2|}{|\sigma^2+\Sigma_{z_n}|^2}\cos(\arg(y_n^*\mu_{z_n})-\mu_{\theta^z_n})\mathbf{R_0}\left(\frac{1}{\Sigma^z_\theta}\right) \tag{37}$$
$$+\frac{\Sigma_{z_n}\sigma^2}{\sigma^2+\Sigma_{z_n}}-E_{Z|Y}\{z_n|y_n,\mu_{z_n},\Sigma_{z_n}\}^2, \tag{38}$$

μ_{θ_n} (resp. Σ_{θ_n}) is the marginalized posterior mean (resp. variance) of the phase noise θ_n as discussed in the next section, and $\mathbf{R_0}(\cdot)=\frac{I_1(\cdot)}{I_0(\cdot)}$ where $I_n(\cdot)$ is the modified Bessel function of the first kind at order n.

Another distribution we have to compute is $p_{x|R}(x|\mu_{x_m},\Sigma_{x_m})$, is the *a-posteriori* distribution of $\mathbf{x}$ regarding propagation of the Gaussian fields propagated by the model, the calculation of the momentum defined as g_{in} and g'_{in} will follow the works presented in [21] to redefine the generic function proposed in SwAMP.

$$E_{X|Y}(x_m|\mu_{x_m},\Sigma_{x_m}) = \frac{\rho\sqrt{2\pi\nu^2}}{Z_{nor}}e^{-\frac{|m_x-\mu_{x_m}|^2}{2(\sigma^2+\Sigma_{x_m})}}\tilde{M} \tag{39}$$
$$var_{X|Y}(x_m|\mu_{x_m},\Sigma_{x_m}) = \frac{\rho\sqrt{2\pi\nu^2}}{Z_{nor}}e^{-\frac{|m_x-\mu_{x_m}|^2}{2(\sigma^2+\Sigma_{x_m})}}|\tilde{M}^2+\nu^2| \quad -E_{X|Y}(x_m|\mu_{x_m},\Sigma_{x_m})^2 \tag{40}$$

with

$$Z_{nor} = \rho\sqrt{2\pi\nu^2}e^{-\frac{|m_x-\mu_{x_m}|^2}{2(\sigma^2+\Sigma_{x_m})}}+(1-\rho)e^{-\frac{|\mu_{x_m}|^2}{2\Sigma_{x_m}}}, \tag{41}$$
$$\tilde{M} = \frac{\sigma^2\mu_{x_m}+\Sigma_{x_m}m_x}{\Sigma_{x_m}+\sigma^2}, \qquad \nu^2=\frac{\sigma^2\Sigma_{x_m}}{\Sigma_{x_m}+\sigma^2}. \tag{42}$$

3.3 Mean-field approximation for the phase noise

The above expressions call on the knowledge of the moments of the posterior distribution on $\boldsymbol{\theta}$. To improve the estimation problem, we use an EM estimation of the prior parameters over $\boldsymbol{\theta}$. This step allow us to maximise the lower bound of the likelihood of $p(\mathbf{y}|\mathbf{x},\boldsymbol{\theta}|)$.In addition, an EM-step will allow us to recover the true parameters of the phase noise prior representing the strength of the fluctuations of the propagation medium. To simplify the latter computation, we propose in this step to resort to a mean-field approximation. Following a similar reasoning as in [12], we get

$$q(\boldsymbol{\theta}) = \mathcal{N}(\boldsymbol{\theta};\boldsymbol{\mu}_\theta,\Sigma_\theta), \tag{43}$$
$$\text{where } \boldsymbol{\mu}_\theta = \Sigma_\theta\left(\text{diag}\left(\frac{2}{\sigma^2}|\boldsymbol{\eta}|\right)\arg(\boldsymbol{\eta})\right), \tag{44}$$
$$\Sigma_\theta^{-1} = \Lambda_\theta^{-1}+\text{diag}\left(\frac{2}{\sigma^2}|\boldsymbol{\eta}|\right) \tag{45}$$

with $\eta_n = y_n\sum_{m=1}^M d^*_{nm}E^*_{X|Y}\{x_m|\mu_{x_m},\Sigma_{x_m}\}$, the nth element in $\boldsymbol{\eta}$ ($|\boldsymbol{\eta}|$ stands here for the element-wise absolute value of $\boldsymbol{\eta}$ and $.^*$ for the complex conjugate), and Λ_θ^{-1} is the precision matrix attached to the prior distribution (4) on $\boldsymbol{\theta}$, *i.e.*

$$\Lambda_\theta^{-1} = \begin{pmatrix} \frac{1}{\sigma_1^2}+\frac{\beta^2}{\sigma_\theta^2} & -\frac{\beta}{\sigma_\theta^2} & 0 & 0 \\ -\frac{\beta}{\sigma_\theta^2} & \frac{1+\beta^2}{\sigma_\theta^2} & \ddots & 0 \\ 0 & \ddots & \ddots & -\frac{\beta}{\sigma_\theta^2} \\ 0 & 0 & -\frac{\beta}{\sigma_\theta^2} & \frac{1}{\sigma_\theta^2} \end{pmatrix}. \tag{46}$$

Note that since the distribution $q(\boldsymbol{\theta})$ is Gaussian, marginals $q(\theta_n)$ used in the previous “prSAMP-step” of the algorithm come as

$$q(\theta_n) = \mathcal{N}(\theta_n; \mu_{\theta_n}, \Sigma_{\theta_n}) \tag{47}$$

where μ_{θ_n} (resp. Σ_{θ_n}) is the nth element in $\boldsymbol{\mu}_\theta$ (resp. in the diagonal of Σ_{θ_n}).

We further study the interest of such an estimation. Indeed, if there are guarantees of convergence for EM extension for additive gaussian noise [21], there is no such guarantees for a structured phase noise. Hence, we will compare and observe the limitations of an EM-prSAMP algorithm.

3.4 Additive noise estimation

In order to refine our estimation we propose to estimate σ^2, the second order momentum of the additive noise, according to the maximum likelihood criterion of the posterior distribution, we finally have to find :

$$\hat{\sigma}^2 = \underset{\sigma^2}{\operatorname{argmax}} \int_{\mathbf{z},\boldsymbol{\theta}} p(\mathbf{z},\boldsymbol{\theta}|\mathbf{y}) \log(p(\mathbf{y},\mathbf{z},\boldsymbol{\theta};\sigma^2)) d\mathbf{z} d\boldsymbol{\theta} \tag{48}$$

According to [12]

$$\hat{\sigma}^2 = \frac{1}{N}\left(\mathbf{y}^H\mathbf{y} - 2\mathcal{R}(\overline{\mathbf{y}}^H\mathbf{E}_{Z|Y}) + \mathbf{E}_{Z|Y}^H\mathbf{E}_{Z|Y} + \mathbf{var}_{Z|Y}\right) \tag{49}$$

with

$$\overline{\mathbf{y}} = \left[y_n e^{-j\mu_{\theta_n}}\mathbf{R}_0\left(\frac{1}{\Sigma_\theta}\right)\right]_{n=\{1,\dots,N\}}, \tag{50}$$
$$\mathbf{E}_{Z|Y} = [E^*_{Z|Y}\{z_1|y_1,\omega_1,\Sigma_{z_1}\}\dots E^*_{Z|Y}\{z_N|y_N,\mu_{z_n},\Sigma_{z_n}\}]^T.$$

In the following, we refer to the proposed two-procedure as “paSAMP” for “phase-aware SwAMP algorithm”. The pseudo-code of paSAMP is presented in **Algorithm 2** with the references to the intermediary variables (μ_{z_n}, Σ_{z_n}, μ_{x_n}, Σ_{x_n}) introduced above.

Algorithm 2 paSAMP Algorithm

Input: $\mathbf{y}$, $\mathbf{D}$, σ^2, ρ, σ_x^2, $\boldsymbol{\mu}_\theta, \Sigma_\theta, T_{max}$

Define according to (36-39):

$g_{out,n} \triangleq \frac{1}{\Sigma_{z_n}}(E_{Z|Y,P}\{z_n|y_n, \mu_{z_n}, \Sigma_{z_n}\} - \mu_{z_n})$

$g'_{out,n} \triangleq \frac{1}{\Sigma_{z_n}}(\frac{(var_{Z|Y,P}\{z_n|y_n, \mu_{z_n}, \Sigma_{z_n}\})}{\Sigma_{z_n}} - 1)$

$g_{in,m} \triangleq E_{X|Y}\{x_m|\mu_{x_m}, \Sigma_{x_m}\}$

$g'_{in,m} \triangleq var_{X|Y}\{x_m|\mu_{x_m}, \Sigma_{x_m}\}$

while $t < T_{max}$ **do**
  for $n = 1 \ldots N$ **do**
    $\hat{z}_n(t) = \sum_{m=1}^{M} d_{nm} a_m(t)$
    $\Sigma_{z_n}^1(t+1) = \sum_{m=1}^{M} |d_{nm}|^2 v_m(t)$
    $\mu_{z_n}^1(t+1) = \hat{z}_n(t) - \Sigma_{z_n}^1(t) g_{out,n}$
  end for
  for $m = permute[1 \ldots M]$ **do**
    $\Sigma_{x_m}(t+1) = (-\sum_{n=1}^{N} |d_{nm}|^2 g'_{out,n})^{-1}$
    $\mu_{x_m}(t+1) = a_m(t) + \Sigma_{x_m}(t+1) \sum_{n=1}^{N} d_{nm} g_{out,n}$
    $v_m(t+1) = \Sigma_{x_m}(t+1) g'_{in,m}$
    $a_m(t+1) = g_{in,m}$
    for $n = 1 \ldots N$ **do**
      $\Sigma_{z_n}^{m+1}(t+1) =$
        $\Sigma_{z_n}^{m}(t+1) + |d_{nm}|^2 (v_m(t+1) - v_m(t))$
      $\mu_{z_n}^{m+1}(t+1) =$
        $\mu_{z_n}^{m}(t+1) + d_{nm}(a_m(t+1) - a_m(t))$
        $-g_{out,n}(t)(\Sigma_{z_n}^{m+1}(t+1) - \Sigma_{z_n}^{m}(t+1))$
    end for
    update σ^2 according to [8].
    if phase noise EM estimation **then**
      update $[\mu_{\theta_n}, \Sigma_{\theta_n}]$ according to (44,45).
    end if
  end for
end while
Output: $\{\hat{x}_m = a_m(T_{max})\}_m$

Once implemented, paSAMP will return $\mathbf{a}$ and $\mathbf{v}$, respectively the mean and variance of $p(\mathbf{x}|\mathbf{y})$, with $\mathbf{a}$ the reconstructed vector $\mathbf{x}$.

4 Numerical Experiments

In this section, we perform a quantitative and qualitative evaluation of the proposed approach with respect to state-of-the-art algorithms.

4.1 First Experiment: Evaluation on the performances of paSAMP without EM noise estimation

We consider the identification problem of $K = 2, 5, 10$ plane waves measured by $N = 256$ sensors on an ULA. We assume that the angles of the K incident waves can be written as $\phi_k = \frac{\pi}{2} + i_k \frac{\pi}{50}$ with $i_k \in [1, 50]$. A set of $M = 50$ steering vectors is defined from a set of angles $\{\phi_i = -\pi + i\frac{\pi}{50}\}_{i \in \{1,\ldots,50\}}$ and the choice of the parameter $\lambda/\Delta = 4$. For each of the K incident waves, the coefficient x_{i_k} is initialized with $m_x = 0.5 + j0.5$, $\rho = K/M$ and $\sigma_x^2 = 0.1$.

We compare the performance of the following 4 different algorithms: *i)* the standard beamforming introduced in [1] (dashed black curve, triangle mark); *ii)* the prSAMP algorithm proposed in [9] as a solution to the phase retrieval problem (continuous black curve, diamond mark); *iii)* the paVBEM procedure proposed in [12] exploiting the same prior models (dashed red curve, circle mark); *iv)* the paSAMP algorithm described in Section 3 (continuous blue curve, square mark).

To evaluate the performance of these procedures, we consider the normalized correlation between the ground truth $\mathbf{x}$ and its reconstruction $\hat{\mathbf{x}}$, that is $\frac{|\mathbf{x}^H \hat{\mathbf{x}}|}{\|\mathbf{x}\|\|\hat{\mathbf{x}}\|}$, as a function of the additive noise variance σ^2. This quantity is averaged over 100 realizations for each point of simulation.

Figures 1(a) to (i) represent the performance of these algorithms for three different phase noises : $\sigma_\theta^2 = 0.03$, $\sigma_\theta^2 = 0.1$ et $\sigma^2 = 0.3$. In those cases, we set the parameters of the Markovian model to $\sigma_0^2 = 10$ and $\beta = 1$, which

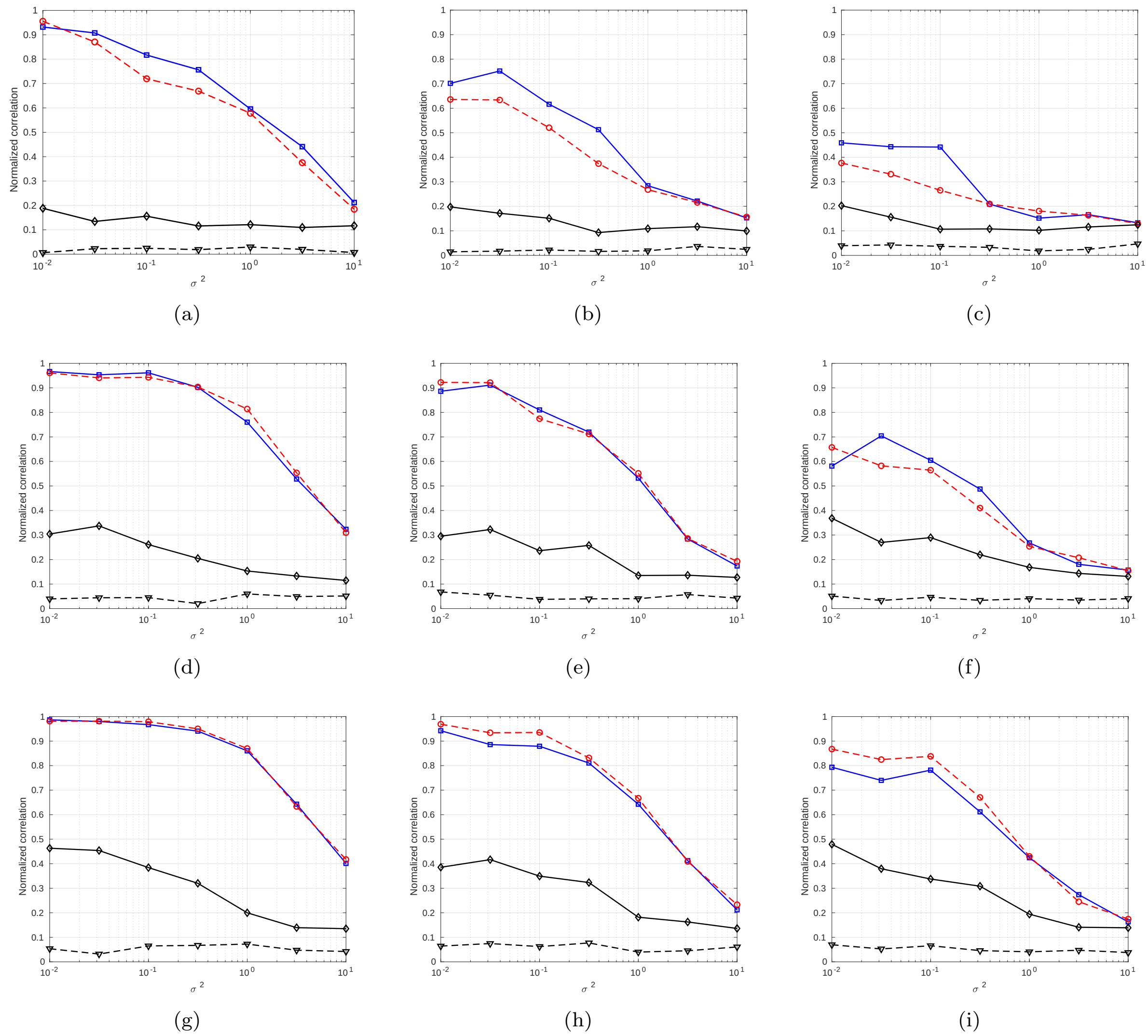

Fig. 1: Normalized correlation regarding the additive noise σ^2 resp. for $\sigma_\theta^2 = 0.03$, $\sigma_\theta^2 = 0.1$ and $\sigma_\theta^2 = 0.3$. the figures (a-c) correspond to the $K = 2$ sources case, (d-f) to the $K = 5$ sources case and (d-f) to the $K = 10$ sources case

corresponds to a great uncertainty on the the initial phase measurement but k a good knowledge of the spatial structure of the phase fluctuations.

Results show that given a known number of sources, both ”phase-aware” algorithms reconstruct the vector $\mathbf{x}$ with more precision than the other algorithms which do not integrate the phase perturbation model. In addition, we notice that paSAMP achieve a better estimation than paVBEM especially in high sparsity cases (low number of active DOA).

However, this gap tends to be reduced, even reversed with the number of active DOA. The deterioration of the performance of paSAMP given the increasing number or sources seems to be due to the “Gaussian Fields”, i.e the gaussian approximation of the Bernouilli-Gaussian distribution over $\mathbf{x}$.

4.2 SECOND Experiment: Evaluation on the performances of paSAMP with Mean-Field phase noise estimation

Still comparing the performance of the algorithms described in the previous section we want to observe the performances with the Mean-Field estimation of the phase noise step described section 3.3.

To evaluate the performance of these procedures, we consider the normalized correlation between the ground truth $\mathbf{x}$ and its reconstruction $\hat{\mathbf{x}}$, that is $\frac{|\mathbf{x}^H\hat{\mathbf{x}}|}{\|\mathbf{x}\|\|\hat{\mathbf{x}}\|}$, as a function of the additive noise variance σ^2. This quantity is averaged over 100 realizations for each point of simulation.

The results achieved by the 4 procedures are presented in Figure 2, resp. for $K = 2$ (left) and $K = 5$ (right) sources. In both cases, we see that the conventional beamforming and the prSAMP algorithm fail to reconstruct $\mathbf{x}$ properly. These resuts illustrate the benefits of carefully accounting for the phase noise in fluctuating environments. We can also notice the superiority of paSAMP over its mean-field counterpart paVBEM, especially in presence of

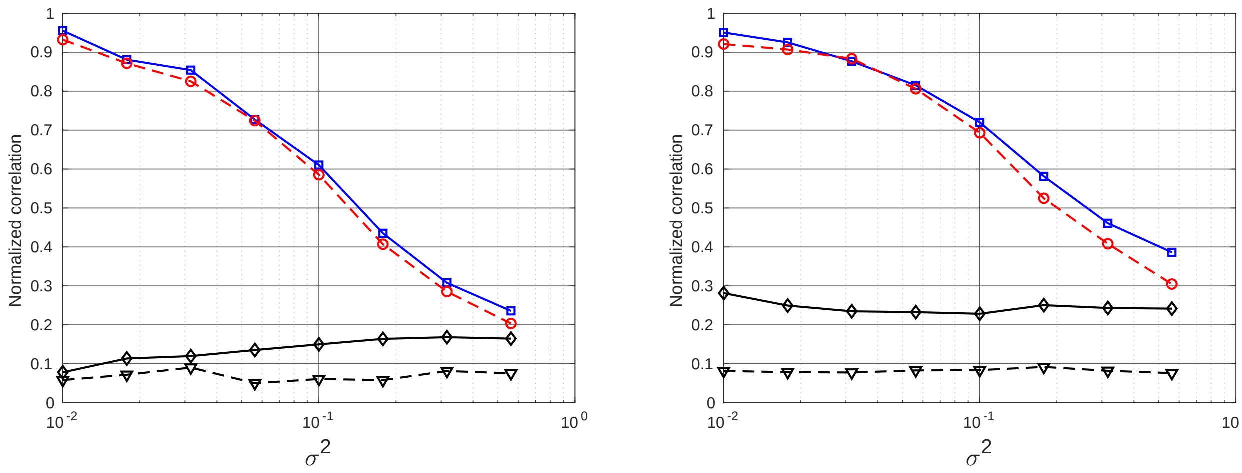

Fig. 2: Evolution of the (averaged) normalized correlation as a function of the variance σ^2 for $K = 2$ (left) and $K = 5$ (right), Comparison of the performance of conventionnal (delay-and-sum) beamforming (triangle mark), "prSAMP" (diamond mark), "paVBEM" (circle mark) and "paSAMP" with EM noise learning(square mark). Experiments show that "paSAMP" provides better results and successfully integrates the phase noisy observation model.

a strong additive noise. This comes in the continuity of previous work [9], where prSAMP proved to outperform prVBEM. Finally, it is interesting to compare the performance of both paSAMP and paVBEM algorithms with regard to the number of sources. Both achieve better performance when confronting to $K = 5$ sources than to $K = 2$ sources. As mentioned in [12], this behavior is typical for the phase retrieval problems, where the loss information on the phases can be compensated by a larger number of sources. In addition, we observe that the performance gap between paSAMP and paVBEM tends to increase with the number of sources. It shows that a Mean-Field estimation step tends to add information about the phase noise and compensate the loss of information due to the second order approximation of the AMP approximation.

4.3 PHASE DIAGRAMS

Here we compare the performances of these algorithms with another representation widely used for phase retrieval algorithms: the phase transition diagrams [9][13]. For these results, we vary the sparse rate $\rho = K/N = 0.2 : 02 : 1$ and the compressing rate $\delta = M/N = 0.2 : 0.2 : 4$. The objective here is to observe the performances of the algorithms in different dimensional setups in order to determine optimal running conditions for future experiments. To do this, we realise X iteration of a dimensional set up with a additive noiseless case $\sigma^2 = 10^{-8}$ and in presence of a phase noise σ_θ with different values from X1 to X2 to evaluate its impact on the estimation performance.

5 CONCLUSION

We have presented here a novel AMP algorithm able to perform DOA estimation in a corrupted phase-noisy environment. This approach exploits both a sparsity prior on the sources and a structured prior on the phase noise. Compared to state-of-the-art algorithms, the approach presents a good behaviour illustrating a successful inclusion of the different assumptions. In particular, it outperforms a recent algorithm dealing with the same DOA estimation problem in fluctuating environments. Future work will include further assessment on real data.

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