Separator Algebra for State Estimation

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1 Introduction

Consider the following state estimation problem [Jau15]

$$\begin{array}{ll} \text{(i)} & \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)), t \in \mathbb{R} \\ \text{(ii)} & \mathbf{g}(t_k) \in \mathbb{Y}(k), \ k \in \mathbb{N} \end{array} \quad (1)$$

Our objective is to find an inner and an outer approximation of the set $\mathbb{X}(t)$ of all state vectors that are consistent with (1) at time t. If we define by flow map φ_{t_1,t_2} as follows:

$$\left(\mathbf{x}(t_1) = \mathbf{x}_1 \text{ and } \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) \Rightarrow \mathbf{x}_2 = \boldsymbol{\varphi}_{t_1,t_2}(\mathbf{x}_1)\right). \quad (2)$$

The set of all *causal feasible states* at time t is defined by

$$\mathbb{X}(t) = \bigcap_{t_k \le t} \boldsymbol{\varphi}_{t_k,t} \circ \mathbf{g}^{-1}(\mathbb{Y}(k)). \quad (3)$$

In this paper, we show how it is possible to find both an inner and an outer approximations for $\mathbb{X}(t)$. Some existing methods are able to find an outer approximation [KJWM99] [GRMA13], but, to my knowledge, none of them is able to get an inner approximation. The main idea is to copy a classical contractor approach [CJ09] for state estimation, but to use separators [JD14] instead of contractors.

2 Separators

In this section, we present separators and show how they can be used by a paver in order to bracket the solution sets. An *interval* of $\mathbb{R}$ is a closed connected set of $\mathbb{R}$. A box $[\mathbf{x}]$ of $\mathbb{R}^n$ is the Cartesian product of n intervals. The set of all boxes of $\mathbb{R}^n$ is denoted by $\mathbb{IR}^n$. A *contractor* $\mathcal{C}$ is an operator $\mathbb{IR}^n \mapsto \mathbb{IR}^n$ such that $\mathcal{C}([\mathbf{x}]) \subset [\mathbf{x}]$ and $[\mathbf{x}] \subset [\mathbf{y}] \Rightarrow \mathcal{C}([\mathbf{x}]) \subset \mathcal{C}([\mathbf{y}])$. A set $\mathbb{S}$ is *consistent* with the contractor $\mathcal{C}$ (we will write $\mathbb{S} \sim \mathcal{C}$) if for all $[\mathbf{x}]$, we have $\mathcal{C}([\mathbf{x}]) \cap \mathbb{S} = [\mathbf{x}] \cap \mathbb{S}$. A *separator* $\mathcal{S}$ is pair of contractors $\{\mathcal{S}^{\text{in}}, \mathcal{S}^{\text{out}}\}$ such that, for all $[\mathbf{x}] \in \mathbb{IR}^n$, we have $\mathcal{S}^{\text{in}}([\mathbf{x}]) \cup \mathcal{S}^{\text{out}}([\mathbf{x}]) = [\mathbf{x}]$. A set $\mathbb{S}$ is *consistent* with the separator $\mathcal{S}$ (we write $\mathbb{S} \sim \mathcal{S}$), if $\mathbb{S} \sim \mathcal{S}^{\text{out}}$ and $\overline{\mathbb{S}} \sim \mathcal{S}^{\text{in}}$. where $\overline{\mathbb{S}} = \{\mathbf{x} \mid \mathbf{x} \notin \mathbb{S}\}$. Using a separator inside a *paver* we can easily to classify part of the search space that are inside or outside a solution set $\mathbb{S}$ associated with $\mathcal{S}$.
The algebra for separators is a direct extension of contractor algebra [CJ09]. If $\mathcal{S}_i = \{\mathcal{S}_i^{\text{in}}, \mathcal{S}_i^{\text{out}}\}, i \in \{1,2\}$ are separators, we define

$$\begin{array}{lll} \mathcal{S}_1 \cap \mathcal{S}_2 &= \{\mathcal{S}_1^{\text{in}} \cup \mathcal{S}_2^{\text{in}}, \mathcal{S}_1^{\text{out}} \cap \mathcal{S}_2^{\text{out}}\} & \text{(intersection)} \\ \mathcal{S}_1 \cup \mathcal{S}_2 &= \{\mathcal{S}_1^{\text{in}} \cap \mathcal{S}_2^{\text{in}}, \mathcal{S}_1^{\text{out}} \cup \mathcal{S}_2^{\text{out}}\} & \text{(union)} \\ \mathbf{f}^{-1}(\mathcal{S}_1) &= \{\mathbf{f}^{-1}(\mathcal{S}_1^{\text{in}}), \mathbf{f}^{-1}(\mathcal{S}_1^{\text{out}})\} & \text{(inverse)} \end{array} \quad (4)$$

If $\mathbb{S}_i$ are sets of $\mathbb{R}^n$, we have [JSD14]

$$\begin{array}{rl} \text{(i)} & \mathbb{S}_1 \cap \mathbb{S}_2 \sim \mathcal{S}_1 \cap \mathcal{S}_2 \\ \text{(ii)} & \mathbb{S}_1 \cup \mathbb{S}_2 \sim \mathcal{S}_1 \cup \mathcal{S}_2 \\ \text{(iii)} & \mathbf{f}^{-1}(\mathbb{S}_1) \sim \mathbf{f}^{-1}(\mathcal{S}_1)\,. \end{array} \tag{5}$$

Interval analysis [Moo66] [KK96] combined with contractors [CJ09] has been shown to be able to

give an outer approximation of set. For the inner subpaving, the *De Morgan* rules make it possible to express the complementary set $\overline{\mathbb{X}}$ of $\mathbb{X}$. Then, basic contractor techniques can be used to get an inner characterization $\mathbb{X}^-$. Now, the task is not so easy and the role of *separators* is to make it automatic.

3 Transformation of separators

A transformation is an invertible function $\mathbf{f}$ such as an analytical expression if known for both $\mathbf{f}$ and $\mathbf{f}^{-1}$. The set of transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ is a group with respect to the composition $\circ$. Symmetries, translations, homotheties, rotations, ... are linear transformations.
Theorem. Consider a set $\mathbb{X}$ and a transformation $\mathbf{f}$. Denote by $[\mathbf{f}]$ and $[\mathbf{f}^{-1}]$ two inclusion functions for $\mathbf{f}$ and $\mathbf{f}^{-1}$. If $\mathcal{S}_{\mathbb{X}}$ is a separator for $\mathbb{X}$ then a separator $\mathcal{S}_{\mathbb{Y}}$ for $\mathbb{Y} = \mathbf{f}(\mathbb{X})$ is

$$[\mathbf{y}] \rightarrow \left\{ \left([\mathbf{f}] \circ \mathcal{S}_{\mathbb{X}}^{\text{in}} \circ \left[\mathbf{f}^{-1}\right]\right)([\mathbf{y}]) \cap [\mathbf{y}]\,, \left([\mathbf{f}] \circ \mathcal{S}_{\mathbb{X}}^{\text{out}} \circ \left[\mathbf{f}^{-1}\right]\right)([\mathbf{y}]) \cap [\mathbf{y}] \right\} \tag{6}$$

or equivalently

$$\mathbf{f}(\mathbb{X}) \sim \left\{ [\mathbf{f}] \circ \mathcal{S}_{\mathbb{X}}^{\text{in}} \circ \left[\mathbf{f}^{-1}\right] \cap \text{Id},\ [\mathbf{f}] \circ \mathcal{S}_{\mathbb{X}}^{\text{out}} \circ \left[\mathbf{f}^{-1}\right] \cap \text{Id} \right\} \tag{7}$$

where *Id* is the identity contractor.
Remark. The separator defined by (6) corresponds to what we call the *transformation* of a separator by $\mathbf{f}$ and we write $\mathcal{S}_{\mathbb{Y}} = \mathbf{f}(\mathcal{S}_{\mathbb{X}})$. As a consequence, thanks to the theorem, we can add to (5) the property

$$\text{(iv)} \quad \mathbf{f}(\mathbb{X}) \sim \mathbf{f}(\mathcal{S}_{\mathbb{X}})\,.$$

which will be used later for our state estimation problem.
Proof. The separator $\mathcal{S}_{\mathbb{Y}}$ is equivalent to $\mathbb{Y} = \mathbf{f}(\mathbb{X})$ if

$$\begin{cases} \mathcal{S}_{\mathbb{Y}}^{\text{out}}([\mathbf{y}]) \cap \mathbb{Y} = [\mathbf{y}] \cap \mathbb{Y} \\ \mathcal{S}_{\mathbb{Y}}^{\text{in}}([\mathbf{y}]) \cap \overline{\mathbb{Y}} \ = [\mathbf{y}] \cap \overline{\mathbb{Y}}. \end{cases} \tag{8}$$

Since $\mathcal{S}_{\mathbb{Y}}^{\text{out}}([\mathbf{y}]) \subset [\mathbf{y}]$ and $\mathcal{S}_{\mathbb{Y}}^{\text{out}}([\mathbf{y}]) \subset [\mathbf{y}]$, it suffices to prove that

$$\begin{cases} \text{(i)} & \mathcal{S}_{\mathbb{Y}}^{\text{out}}([\mathbf{y}]) \supset [\mathbf{y}] \cap \mathbb{Y} \\ \text{(ii)} & \mathcal{S}_{\mathbb{Y}}^{\text{in}}([\mathbf{y}]) \ \supset [\mathbf{y}] \cap \overline{\mathbb{Y}}. \end{cases} \tag{9}$$

Let us first prove (i). We have

$$\begin{array}{rll} [\mathbf{y}] \cap \mathbb{Y} & = \mathbf{f}\left(\mathbf{f}^{-1}([\mathbf{y}]) \cap \mathbf{f}^{-1}(\mathbb{Y})\right) & \mathbf{f} \text{ is bijective} \\ & = \mathbf{f}\left(\mathbf{f}^{-1}([\mathbf{y}]) \cap \mathbb{X}\right) & \mathbb{X} = \mathbf{f}^{-1}(\mathbb{Y}) \\ & \subset \mathbf{f}\left(\left[\mathbf{f}^{-1}\right]([\mathbf{y}]) \cap \mathbb{X}\right) & \left[\mathbf{f}^{-1}\right] \text{ is an inclusion function for } \mathbf{f}^{-1} \\ & \subset \mathbf{f}(\mathcal{S}_{\mathbb{X}}^{\text{out}}\left(\left[\mathbf{f}^{-1}\right]([\mathbf{y}])\right)) & \mathcal{S}_{\mathbb{X}}^{\text{out}} \text{ is a contractor for } \mathbb{X} \\ & \subset [\mathbf{f}] \circ \mathcal{S}_{\mathbb{X}}^{\text{out}} \circ \left[\mathbf{f}^{-1}\right]([\mathbf{y}]) & [\mathbf{f}] \text{ is an inclusion function for } \mathbf{f} \end{array} \tag{10}$$

Thus $[\mathbf{y}] \cap \mathbb{Y} \subset \left([\mathbf{f}] \circ \mathcal{S}_{\mathbb{X}}^{\text{out}} \circ \left[\mathbf{f}^{-1}\right]([\mathbf{y}]) \cap [\mathbf{y}]\right) = \mathcal{S}_{\mathbb{Y}}^{\text{out}}([\mathbf{y}])$. Let us now prove (ii). We have

$$\begin{array}{lll} [\mathbf{y}] \cap \overline{\mathbb{Y}} & = \mathbf{f}\left(\mathbf{f}^{-1}([\mathbf{y}]) \cap \mathbf{f}^{-1}\left(\overline{\mathbb{Y}}\right)\right) & \mathbf{f} \text{ is bijective} \\ & = \mathbf{f}\left(\mathbf{f}^{-1}([\mathbf{y}]) \cap \overline{\mathbb{X}}\right) & \overline{\mathbb{X}} = \mathbf{f}^{-1}\left(\overline{\mathbb{Y}}\right) \\ & \subset \mathbf{f}\left(\left[\mathbf{f}^{-1}\right]([\mathbf{y}]) \cap \overline{\mathbb{X}}\right) & \left[\mathbf{f}^{-1}\right] \text{ is an inclusion function for } \mathbf{f}^{-1} \\ & \subset \mathbf{f}(\mathcal{S}_{\mathbb{X}}^{\text{in}}\left(\left[\mathbf{f}^{-1}\right]([\mathbf{y}])\right)) & \mathcal{S}_{\mathbb{X}}^{\text{in}} \text{ is a contractor for } \overline{\mathbb{X}} \\ & \subset [\mathbf{f}] \circ \mathcal{S}_{\mathbb{X}}^{\text{in}} \circ \left[\mathbf{f}^{-1}\right]([\mathbf{y}]) & [\mathbf{f}] \text{ is an inclusion function for } \mathbf{f} \end{array} \quad (11)$$

Thus $[\mathbf{y}] \cap \mathbb{Y} \subset \left([\mathbf{f}] \circ \mathcal{S}_{\mathbb{X}}^{\text{out}} \circ \left[\mathbf{f}^{-1}\right]([\mathbf{y}]) \cap [\mathbf{y}]\right) \cap \mathbb{Y} = \mathcal{S}_{\mathbb{Y}}^{\text{in}}([\mathbf{y}])$ which terminates the proof. ■

Example. Consider the constraint

$$\left\| \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} y_1 - 1 \\ y_2 - 2 \end{pmatrix} \right\| \in [1, 3]. \quad (12)$$

If we apply an efficient forward-backward contractor in a paver, we get the contractions illustrated by the paving of Figure 1, left. Now, if we take

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} y_1 - 1 \\ y_2 - 2 \end{pmatrix} = \mathbf{f}^{-1}(\mathbf{y}) \quad (13)$$

or equivalently

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \mathbf{f}(\mathbf{x}), \quad (14)$$

we get

$$\mathbf{y} = \mathbf{f}(\mathbf{x}), \text{ and } \|\mathbf{x}\| \in [1, 3]. \quad (15)$$

An optimal separator $\mathcal{S}_{\mathbb{X}}$ can be built for $\mathbf{x}$ and the separator transform provides us a separator $\mathcal{S}_{\mathbb{Y}}$ for $\mathbb{Y}$. As illustrated by Figure 1, right, the resulting separator $\mathcal{S}_{\mathbb{Y}}$ is more efficient than the classical one based on forward-backward contractors. Note that in case we are not able to have an inner approximation for $\mathbf{f}^{-1}$, the problem of finding an inner approximation of the image of a set $\mathbf{f}(\mathbb{X})$ becomes much more difficult. See, *e.g.*, [VJVS05] [GJ10].

4 State estimation

If $\mathcal{S}_{\mathbb{X}(0)}$ is a separator for $\mathbb{X}(0)$ and if $\mathcal{S}_{\mathbb{Y}(k)}$ are separators for $\mathbb{Y}(k)$, then a separator for the set $\mathbb{X}(t)$ defined by (3) is

$$\mathcal{S}_{\mathbb{X}(t)} = \bigcap_{t_k \leq t} \boldsymbol{\varphi}_{t_k,t} \circ \mathbf{g}^{-1}\left(\mathcal{S}_{\mathbb{Y}(k)}\right). \quad (16)$$

In this formula, $\mathbf{g}^{-1}\left(\mathcal{S}_{\mathbb{Y}(k)}\right)$ is a separator. Due to the fact that $\boldsymbol{\varphi}_{t_k,t}$ is bijective and that we are able to find an inclusion function for $\boldsymbol{\varphi}_{t_k,t}$ and $\boldsymbol{\varphi}_{t_k,t}^{-1}$ [RN11], the separator $\boldsymbol{\varphi}_{t_k,t} \circ \mathbf{g}^{-1}\left(\mathcal{S}_{\mathbb{Y}(k)}\right)$ is clearly defined using the separator transform. To illustrate the method, let us consider a robot described by

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{pmatrix} v(t)\cos\theta(t) \\ v(t)\sin\theta(t) \end{pmatrix} & \text{(evolution)} \\ \|\mathbf{x}(t_k)\| \in y(t_k) + [-0.3, 0.3], \ t_k = 0.1 \cdot k, \ \ k \in \mathbb{N} & \text{(observation)} \end{cases} \quad (17)$$

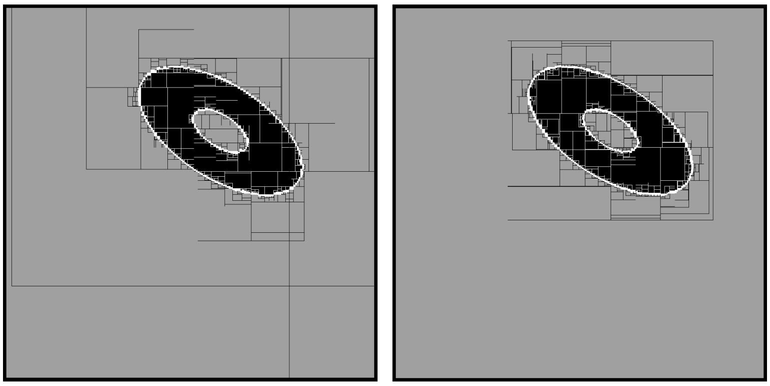

Fig. 1. Left. Contractions obtained using a classical forward-backward propagation; Right. Contractions obtained using the separator transform. The frame corresponds to the box $[-6,6]^2$.

where $v(t)$ and $\theta(t)$ are measured with an accuracy of ± 0.03. The observation equation is due to the fact that the robot measures every 0.1 sec its distance to the origin with an accuracy of ± 0.3. The actual (but unknown) trajectory for the robot is

$$\mathbf{x}(t) = \begin{pmatrix} 2 + 3\cos t \\ 2\sin t \end{pmatrix}. \tag{18}$$

For $t \in 0.2 * k,\ k = 0, \ldots, 7$, the sets $\mathbb{X}(t)$ obtained by our observer are represented on Figure 2. Black boxes are inside $\mathbb{X}(t)$, grey boxes are outside and the white boxes cover the boundary. For $t = 0$, $\mathbb{X}(t)$ is a ring which becomes a small set for $t = 1.4$ once the robot has moved sufficiently. The fact that the white area covering the boundary becomes thick is mainly due to the state errors inside the evolution equation.

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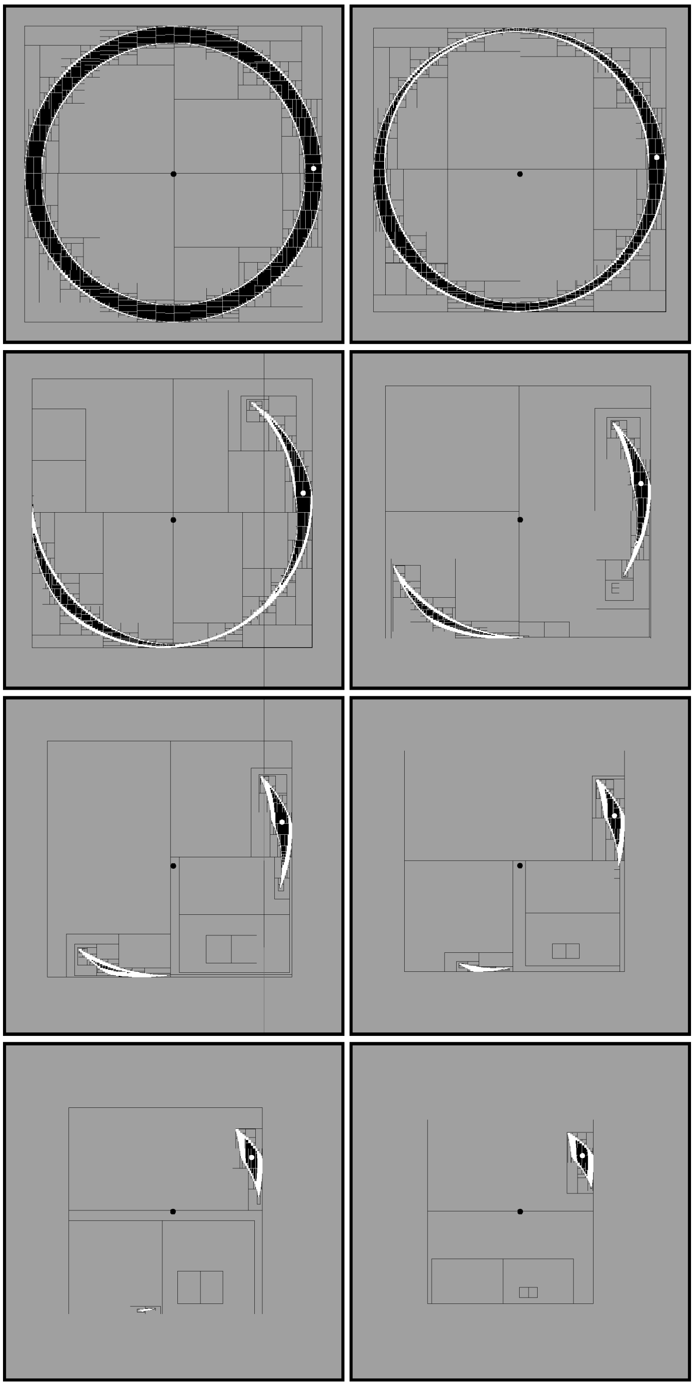

Fig. 2. Inner and outer approximations of the set of all feasible state vectors $\mathbb{X}(t)$, for $t \in 0, 0.2, \ldots, 1.4$. The frame boxes are $[-6, 6]^2$.

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